# Side Project: Overview

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This side project is currently on hold, and I anticipate it will remain so for an extended period, as I cannot think of its real-world application.

- (Item1) Consider an election with four candidates, denoted as $\{1, 2, 3, 4\}$, and five voters, denoted as $\{i, j, k, m, l\}$. The voting mechanism employed is a majority rule, represented as $g_{maj}$.
- (Item2) Assume that only *four* types of preferences exist as outlined below.
  - $(1 \succ 2 \succ 3 \succ 4)$: 13%
  - $(2 \succ 1 \succ 3 \succ 4)$: 17%
  - $(3 \succ 2 \succ 1 \succ 4)$: 40%
  - $(4 \succ 2 \succ 3 \succ 1)$: 30%

  The number associated with each preference represents its probability mass.
- (Item3) Given Item 2, let the set of preference profiles be denoted by $\mathcal{S}$. Since there are five residents, the elements of $\mathcal{S}$ are as follows:

$$\begin{aligned} s_1 &= \{(1 \succ 2 \succ 3 \succ 4), (1 \succ 2 \succ 3 \succ 4), (1 \succ 2 \succ 3 \succ 4), (1 \succ 2 \succ 3 \succ 4), (1 \succ 2 \succ 3 \succ 4)\} \\ s_2 &= \{(1 \succ 2 \succ 3 \succ 4), (1 \succ 2 \succ 3 \succ 4), (1 \succ 2 \succ 3 \succ 4), (1 \succ 2 \succ 3 \succ 4), (4 \succ 2 \succ 3 \succ 1)\} \\ &.... \\ s_{1024} &= \{(4 \succ 2 \succ 3 \succ 1), (4 \succ 2 \succ 3 \succ 1), (4 \succ 2 \succ 3 \succ 1), (4 \succ 2 \succ 3 \succ 1), (4 \succ 2 \succ 3 \succ 1)\} \end{aligned}$$

  The number 1024 comes from the expression $4 * 4 * 4 * 4 * 4$.
- (Item4) For example, say $s_{377}$ is as follows.
  - $i$ has $(3 \succ 2 \succ 1 \succ 4)$
  - $j$ has $(3 \succ 2 \succ 1 \succ 4)$
  - $k$ has $(2 \succ 1 \succ 3 \succ 4)$
  - $m$ has $(1 \succ 2 \succ 3 \succ 4)$
  - $l$ has $(4 \succ 2 \succ 3 \succ 1)$

  Recall that $g_{maj}$ is a majority voting rule, described as follows:
  - Each resident casts a vote by writing the name of one candidate on a piece of paper. The candidate who receives the most votes wins the election. In the event of a tie, the candidate with the higher number is declared the winner (for example, if both candidates 1 and 3 receive two votes, candidate 3 wins because 3 is numerically greater than 1).
- (Item5) I will assume a complete information setting in this case; thus, the residents know they are in $s_{377}$, while the mechanism designer is unaware of the state.

  Given $g_{maj}$, one of the Nash Equilibrium Strategy profiles is $(3, 3, 2, 2, 2)$, meaning that residents $i$, $j$, $k$, $m$, and $l$ votes for candidates 3, 3, 2, 2, and 2 — I will explain why.
  - From $i$'s perspective, the strategy profile he faces is (3,2,2,2). Whether $i$ votes for candidates 1, 2, 3, or 4 does not affect the outcome of the election.
  - $j$'s situation is identical to $i$'s.

- From $k$'s perspective, the strategy profile he faces is (3,3,2,2). If $k$ votes for candidate 2, then candidate 2 wins the election, which is the best outcome for him.
- From $m$'s perspective, the strategy profile he faces is $(3, 3, 2, 2)$. If $m$ votes for candidates 1, 3, or 4, the outcome will be candidate 3 winning. In contrast, if he votes for candidate 2, the outcome will be a victory for candidate 2. Since $m$ prefers candidate 2 over candidate 3, he votes for candidate 2.
- From $l$'s perspective, the profile he faces is (3,3,2,2). If $l$ votes for 4, 3 or 1, the outcome will be candidate 3. In contrast, if he votes for candidate 2, the outcome will be a victory for candidate 2. Since $l$ prefers candidate 2 over candidate 3, he votes for candidate 2.

Let $\sigma^*(\cdot; g_{maj})$ denote the Nash equilibrium strategy profile under the mechanism $g_{maj}$. I conclude that $g_{maj} \circ \sigma^*(s_{377}; g_{maj}) = 2$. For now, I avoid addressing the issue of multiple equilibria and use $\sigma^*(\cdot)$ instead of $\sigma^*(\cdot; g_{maj})$ for simplicity of notation.

- (Item6) Assume that the mechanism designer wants to choose a candidate who is the most popular among the residents' top choices (in $s_{377}$, this is candidate 3). I denote this preference as $f(\cdot)$, so I can write $f(s_{377}) = 3$.

  We observe that $f(s_{377}) \neq g_{maj} \circ \sigma^*(s_{377})$, meaning that $g_{maj}$, which was designed by the mechanism designer, fails to implement his desired $f(\cdot)$, at least in state $s_{377}$. This failure occurs because $f(\cdot)$ is a nonmonotonic social choice function. However, in other states, such as $s_{122}$ or $s_{35}$, we may have $f(s_{122}) = g_{maj} \circ \sigma^*(s_{122})$ and $f(s_{35}) = g_{maj} \circ \sigma^*(s_{35})$.

- (Item7) Now, recall the probability mass function from Item 2. Under this function, the probability of $s_{177}$ may be higher than that of, say, $s_{199}$. In other words, the mass function in Item 2 assigns a probability to each element $s_1, s_2, \ldots, s_{1024}$ in $\mathcal{S}$. I will use the notation $\Pr(\cdot)$ to represent the probability mass function for $\mathcal{S}$.

  As is typically assumed in the textbook case, the mechanism designer knows $\Pr(\cdot)$. Given this, under $\Pr(\cdot)$, the designer can proceed as follows:

  - Draw $X_1$ from $\Pr(\cdot)$. $X_1$ could be $s_1$, $s_{177}$, $s_{199}$, or any other element of $\mathcal{S}$, but we treat it as a random variable. For each draw, obtain the values $f(x_1)$ and $g_{maj} \circ \sigma^*(x_1)$, and keep those in mind.
  - Draw $X_2$ from $\Pr(\cdot)$, and perform the same procedure: obtain both $f(x_2)$ and $g_{maj} \circ \sigma^*(x_2)$, and keep those in mind.

  If the designer repeats this process 1,000 times, he will obtain the pairs $(f(x_1),\ g_{maj} \circ \sigma^*(x_1))$,... $(f(x_{1,000}),\ g_{maj} \circ \sigma^*(x_{1,000}))$.

- (Item 8) For this mechanism, $g_{maj}$, it may be the case that only 78% of the time the mechanism correctly implements the social choice function. However, if we switch to alternative mechanism—such as using a roulette to select a candidate, or conducting an auction—the success rate might increase (but can never be 100% because $f$ is nonmonotonic). If one of the alternatives does improve the success rate, we can compare the existing $g_{maj}$ with the new alternative using a test statistic to determine whether the difference is statistically significant. Some variant of the central limit theorem would be required for this comparison.

- (Item 9) Alternatively, we could fix the mechanism $g_{maj}$ and vary the social choice functions, such as $f(\cdot)$, $\tilde{f}(\cdot)$, and $\tilde{\tilde{f}}(\cdot)$, to see whether $g_{maj}$ performs well across different social choice functions. This statistical approach might help address questions that are difficult to solve purely through theoretical approach.