Disentangling Affiliation and Synergy in First-Price Auctions Under Limited Disclosure*

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Abstract

I consider a two-period first-price auction where the auctioneer sells a single unit each period, and discloses only the winner's identity between the periods. If a bidder wins both auctions, either (a) the first unit makes the second unit more valuable(*synergy*) or (b) the first unit has no causal effect(*no synergy*) but is a byproduct of a bidder highly valuing both units(*affiliation*); the presence of *synergy* entails different auction design, such as whether to bundle both units or not. Under the independent private value paradigm, I develop a model that treats synergy and affiliation separately. For the separation, I use a nonparametric identification strategy; the strategy is also applied to making the kernel density estimator whose simulation result shows its accuracy.

Keywords: sequential first-price auction, nonparametric, kernel estimation

JEL Codes: C14, C51, C57

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1 Introduction

Consider two agents, where one agent experiences an event more frequently than the other. In the context of an auction, this repeated experience is analogous to the winner of the current auction being more likely to win subsequent auctions.

One possible explanation for the repeated winning is that owning the current object enhances the preference for the following object, indicating a synergy between the two. Knowing the degree of synergy helps the auctioneer's decision: the FCC's spectrum auction separates licenses into groups where those with high synergy are located in the same group, while across groups, licenses have minimal synergy.[1] Another explanation for the repeated wins is that bidders have different preferences for the objects, and these differences persist in subsequent auctions; in this case, the same bidder wins many auctions even without the presence of synergy. Without confirming the degree of synergy, the auctioneer may erroneously believe in the first explanation when the truth is the second explanation, known as the spurious state dependence (Heckman (1981)).

Repeated winning occurs when a bidder wants more than one object, a situation known as multi-unit demand. Papers assuming a multi-unit demand have mostly focused on auctions other than the first-price sealed-bid: these include second-price sealed-bid auctions (Katzman (1999), Liu (2021)), English auctions (Branco (1996), Donna and Espín-Sánchez (2018)), and a combination of English and first-price sealed-bid auctions (Kong (2021)).

Jofre-Bonet and Pesendorfer (2003) is among the studies that focuses on repeated first-price sealed-bid auctions. In their paper, a bidder[2] wants multiple projects, but previously won projects causes him to bid less aggressively. The causal effect of owning one object on the value of another is also modeled in Arsenault-Morin et al. (2022), Gentry et al. (2023), and Altmann (2024). Each of the three models assumes that this causal effect, or synergy between the objects, is solely determined by the characteristics of the objects. This sole determination does not allow the synergy between the objects to vary based on how much a bidder values them; in my model, I use a function δ to allow this variation.

I examine two-period first-price sealed-bid auctions in which single object is auctioned off each period. Between the two periods, the auctioneer discloses only the first auction winner's identity to the bidders, without revealing the winning bid. This limited disclosure and its effect on a bidder's bidding strategy have been discussed in Bergemann and Hörner (2018). Their paper and my model both assume a bidder's multi-unit demand, but I use Perfect Bayesian Nash Equilibrium instead of Markov Perfect Equilibrium.

In my model, a bidder's value for the first object influences both the value and the likelihood of his valuation for the second object through $F_{2|1}$. When the values for both objects highly correlated for him, his value for the second object closely mirrors his value for the first object. This high correlation, captured by $F_{2|1}$, is distinct from how his value for the second object changes due to his owning the first object, represented by the function δ (Section 2). This distinction helps the auctioneer accurately assess the auction at hand. Since equilibrium strategies are derived by restricting the possible set of bid distributions, not every distribution can be

[1] See Subsections C and D in Federal Communications Commission (1994).

[2] One of the *Regular* bidders in their case, not *Fringe* bidders.

rationalized by my model. Section 3 demonstrates that both δ and $F_{2|1}$ can be identified with limited observations: the winner's identity and the winning bid are sufficient. The case where all bids and bidders are observed is a special case. Section 4 details the multi-step estimator, and Section 5 confirms the estimator's reliable performance through Monte Carlo simulations.

2 Model

I present the framework in 2.1 and introduce the Bayesian Nash equilibrium strategies in 2.2.

2.1 Framework

The setting is the same as in Kong (2021) except that I consider a sequence of two first-price sealed-bid auctions — letters in uppercase and lowercase stand for random variable and realized value.

w, l:	First auction winner(w), first auction loser(l).
$V_1, F_1(z)$:	Value of the first object(V_1) and its distribution($F_1(z) \equiv \Pr[V_1 \leq z]$).
$V_2, F_{2\|1}(x\|z)$:	V_2 represents the value of the second object without the presence of the first object; it corresponds to the value of the second item for the first auction loser(l). Consequently, $F_{2\|1}(x\|z) \equiv \Pr[V_2 \leq x\|V_1 = z]$ signifies the distribution of the second object's value for the first auction loser(l) when having drawn $V_1 = z$.
$\delta(V_1, V_2), D(d\|z)$:	Consider a scenario where $V_2 = x$ is given. $\delta(V_1 = z, V_2 = x)$ represents the value of the second object when the first auction winner(w) possesses $V_1 = z$ worth of the first object. It reflects the value assigned by the first auction winner(w) to the second item. Given $\delta(V_1, V_2)$, I define $D(d\|z) \equiv \Pr[\delta(V_1, V_2) \leq d\|V_1 = z]$ as the distribution of the second object's value for the first auction winner(w) who holds $V_1 = z$ worth of the first object.
$s_1(V_1)$:	Bayesian Nash equilibrium bidding strategy in the first auction; a subscript 1 of s_1 refers to the first auction.
$s_2^w(V_1, \delta(V_1, V_2))$:	First auction winner(w)'s equilibrium[3] bidding strategy in the second auction; a subscript 2 of s_2^w refers to the second auction.
$s_2^l(V_1, V_2)$:	First auction loser(l)'s equilibrium bidding strategy in the second auction.

Synergy exists when the adjusted value of the second object from having won the first object($\delta(V_1, V_2)$) exceeds the value of the second object by itself(V_2). *Affiliation* exists when a bidder likely draws a higher v_2 the more he values the first object; namely, the conditional distribution of V_2 given V_1 stochastically dominates the other with a lower V_1. Unlike in Milgrom

[3] *equilibrium* means *Bayesian Nash equilibrium* unless otherwise discussed.

and Weber (1982), affiliation is the relationship between the two objects, not the relationship between the bidders' valuation of one object.

Notations for the bids also follow.

B_1, B_2^w, B_2^l:	First auction bid(B_1); second auction bid of the first auction winner(B_2^w); second auction bid of the first auction loser(B_2^l).
$G_1(x)$:	$\Pr[B_1 \leq x]$; the distribution of the first auction bid.
$G_{2\vert 1}^w(x\vert y)$:	$\Pr[B_2^w \leq x\vert B_1 = y]$; the distribution of second auction bid for the first auction winner(w) given that he won the first auction with $B_1 = y$.
$G_{2\vert 1}^l(x\vert y)$:	$\Pr[B_2^l \leq x\vert B_1 = y]$; the distribution of second auction bid for the first auction loser(l) given that he lost the first auction with $B_1 = y$.
$G_2^l(x\vert B_1 \leq y)$:	$\Pr[B_2^l \leq x\vert B_1 \leq y]$; the distribution of the second auction bid for the first auction loser(l) given that his bid B_1 was less than y.

Auction pair indicates the first and second auctions jointly. Let L be the total number of auction pairs that the analyst observes; then, I can always find the *ℓ-th auction pair* for any $\ell \in \{1, \ldots, L\}$.

Under private values assumption, any bidder i in any ℓ-th auction pair follows the *steps* (0)-(iii).

step(0) The auctioneer determines the extent of information disclosure about the first auction that he will provide to the bidders after its conclusion — in our model, he chooses to disclose only the winner's identity, keeping the winning bid and the losing bids confidential.

step(i) i draws v_{1i} from F_1 and places a bid, $s_1(v_{1i})$.

step(ii) The auctioneer concludes the first auction and announces the information based on the chosen policy in *step*(0). Each bidder learns whether he has won or lost in the first auction without the knowledge of the other bidders' bids.

step(iii) i draws v_{2i} from $F_{2|1}(\cdot|v_{1i})$. Given the draw of v_{2i}, if he is the first auction winner he values the second object at $\delta(v_{1i}, v_{2i})$ and places a bid $s_2^w(v_{1i}, \delta(v_{1i}, v_{2i}))$;

v_{1i} and v_{2i} are not drawn at the same step, which follows the spirit of Kong (2021); v_{1i} is fixed at *step*(i), and it influences the distribution of V_2 in *step*(iii).

I use Assumptions 1-5.

Assumption 1 *(No Dropout) Given any ℓ-th auction pair from $\ell \in \{1, \ldots, L\}$, the set of bidders remains the same across the first and the second auction (No dropout).*

Assumption 2 *(No endogenous participation across auction pairs) The set of bidders varies exogenously across elements in $\{1, \ldots, L\}$.*

Assumption 3 *(Independence[4] across auction pairs) Pick arbitrary two elements, say (a, b), from $\{1, \ldots, L\}^2$. Any result from the a-th auction pair has no effect on bidders' valuations and their strategies in the b-th auction pair and vice-versa.*

Assumption 4 *δ is a non-stochastic function from a set $\mathbb{R}_+^2$ to $\mathbb{R}_+$, which is increasing[5] in V_2 for every V_1.*

Assumptions 1-4 render the identification and estimation of the model primitives, $[F_1,\ F_{2|1},\ \delta]$, tractable. Among the assumptions, the Assumption 4 implies that if the value of the second object increases in the absence of the first object, then the adjusted value of the second object can never be diminished from owning the first object.

Assumption 5 discusses the independence of valuations across bidders.

Assumption 5 *(Independence of valuations across bidders) Pick arbitrary $\ell \in \{1, \ldots, L\}$ and denote the number of bidders as I_ℓ. The values $V_{1j}, j = 1, \ldots, I_\ell$ are independent and identically distributed according to F_1; the pairs $(V_{1j}, V_{2j}), j = 1, \ldots, I_\ell$ are independent with a joint density given by $f(v_{11}, v_{21}, \ldots, v_{1I_\ell}, v_{2I_\ell}) = \prod_{j=1}^{I_\ell} f(v_{1j}, v_{2j})$.*

Within a single bidder $j \in \{1, \ldots, I_\ell\}$, V_{2j} and V_{1j} are dependent through $F_{2|1}$ as expressed in *step*(iii); across the bidders, the pairs (V_1, V_2) are independent by Assumption 5, but the second auction bids $(B_{2j}, j = 1, \ldots, I_\ell)$ are not necessarily independent as described in Remark 1.

Remark 1 *The equilibrium bids $s_1(V_{1j}), j = 1, \ldots, I_\ell$ in the first auction are independent and identically distributed as $\Pr[s_1(V_1) \leq \cdot]$. Without loss of generality let bidder k be the winner of the first auction. The equilibrium bids $s_2^w(V_{1k}, \delta(V_{1k}, V_{2k})), s_2^l(V_{1j}, V_{2j}), j \neq k$ in the second auction are not necessarily independent.*

The intuition behind the second auction bids not necessarily being independent is that a bidder i becoming w or l in the second auction depends on other bidders' $V_{1j}, j \neq i$. Instead, if I condition on the first auction's winning bid and winner's identity, the same second auction bids become independent (Lemma 1).

Lemma 1 *Let the first auction winner be a bidder i. The I_ℓ second auction bids are independent conditional on $\{W_1 = i, B_1^{\max} = b_{1i}\} = \{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\}$. The distribution of B_{2i} given $\{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\}$ is $G_{2|1}^w(\cdot|b_{1i})$ whereas for $j \neq i$ the distribution of B_{2j} given $\{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\}$ is $G_2^l(\cdot|B_1 \leq b_{1i})$.*

Lastly, I assume in Sections 3 and 4 that the observed bids result from the equilibrium play, so that $b_1 = s_1$, $b_2^w = s_2^w$, and $b_2^l = s_2^l$ hold.

2.2 Equilibrium Strategies

I derive equilibrium strategies, $[s_1, s_2^w, s_2^l]$, such that each strategy is strictly monotone, i.e., s_1 is increasing in V_1 and $\left[s_2^w, s_2^l\right]$ are increasing in V_2 for every V_1.

[4]'independence' means 'mutual independence' unless otherwise discussed.

[5]'increasing' is equivalent to 'strictly increasing' unless otherwise discussed.

Consider a bidder i with valuations (v_{1i}, v_{2i}) who has to choose b_{1i} in the first auction and b_{2i}^{w}(resp., b_{2i}^{l}) in the second auction if he wins(resp., loses) the first auction — the bids $[b_{1i}, b_{2i}^{w}, b_{2i}^{l}]$ need not be the equilibrium bids $\left[s_1(v_{1i}), s_2^{w}(v_{1i}, \delta(v_{1i}, v_{2i})), s_2^{l}(v_{1i}, v_{2i})\right]$. Assume that bidder i's competitors follow the equilibrium strategies so that $[B_{1j} = s_1, B_{2j}^{w} = s_2^{w}, B_{2j}^{l} = s_2^{l}, j \neq i]$, holds.

Let the number of bidders be I, and according to Assumption 1 the set of bidders remains the same in both the first and second auctions. To characterize the equilibrium strategies I reason backward and consider the second auction, which is an asymmetric first-price sealed-bid auction distinguishing whether bidder i won or lost the first auction.

When i is the First Auction Winner: At the start of the second auction or *step*(iii) he only knows that the highest competing bid in the first auction, $B_{1,-i}^{\max}$, was smaller than his winning bid b_{1i}, i.e., $B_{1,-i}^{\max} \leq b_{1i}$. Given $\left\{B_{1,-i}^{\max} \leq b_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}\right\}$ the distribution of the highest competing bid $B_{2,-i}^{\max}$ in the second auction is,

$$\begin{aligned} H_2^{w}(\cdot; b_{1i}) &\equiv \Pr\left[B_{2,-i}^{\max} \leq \cdot \mid B_{1,-i}^{\max} \leq b_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}\right] \\ &= \Pr\left[s_2^{l}(V_{1j}, V_{2j}) \leq \cdot, j \neq i \mid s_1(V_{1j}) \leq b_{1i}, j \neq i, V_{1i} = v_{1i}, V_{2i} = v_{2i}\right] \\ &= \Pr\left[s_2^{l}(V_{1j}, V_{2j}) \leq \cdot, j \neq i \mid s_1(V_{1j}) \leq b_{1i}, j \neq i\right] \\ &= \prod_{j \neq i} \Pr\left[B_{2j} \leq \cdot \mid B_{1j} \leq b_{1i}\right] \equiv G_2^{l}(\cdot \mid B_1 \leq b_{1i})^{I-1}, \end{aligned} \tag{1}$$

where the third and last rows in (1) hold by Assumption 5.[6] Using $H_2^{w}(\cdot; b_{1i})$ the expected profit from choosing b_{2i}^{w} in the second auction is $\pi_2^{w}(v_{1i}, v_{2i}, b_{1i}, b_{2i}^{w}) \equiv [\delta(v_{1i}, v_{2i}) - b_{2i}^{w}]\, H_2^{w}(b_{2i}^{w}; b_{1i})$ given (v_{1i}, v_{2i}, b_{1i}) are fixed. The first-order condition of $\pi_2^{w}(v_{1i}, v_{2i}, b_{1i}, b_{2i}^{w})$ with respect to b_{2i}^{w} is,

$$\delta(v_{1i}, v_{2i}) = b_{2i}^{w} + \frac{H_2^{w}(b_{2i}^{w}; b_{1i})}{h_2^{w}(b_{2i}^{w}; b_{1i})} \equiv \xi_2^{w}(b_{1i}, b_{2i}^{w}). \tag{2}$$

Assume that $\xi_2^{w}(b_{1i}, b_{2i}^{w})$ is increasing in b_{2i}^{w} so that by Lemma 2 I have a unique solution denoted as $\tilde{b}_{2i}^{w} \equiv \tilde{s}_2^{w}(v_{1i}, \delta(v_{1i}, v_{2i}), b_{1i})$. Using $\tilde{b}_{2i}^{w}$, the expected profit in the second auction upon winning the first auction is,

$$\tilde{\pi}_2^{w}(v_{1i}, v_{2i}, b_{1i}) = H_2^{w}\left(\tilde{b}_{2i}^{w}; b_{1i}\right)^2 / h_2^{w}\left(\tilde{b}_{2i}^{w}; b_{1i}\right).$$

A bidder i uses $\tilde{\pi}_2^{w}(v_{1i}, v_{2i}, b_{1i})$ at *step*(i). Assume that he has drawn v_{1i} and made a bid b_{1i} and is unaware of the outcome of the first auction. By the unawareness the continuation value for winning the first auction at *step*(i) relies on $\tilde{\pi}_2^{w}(v_{1i}, v_{2i}, b_{1i})$ as follows.

$$\begin{aligned} \mathcal{V}^{w}(v_{1i}, b_{1i}) &\equiv \mathbb{E}_{V_2|V_1}\left[\tilde{\pi}_2^{w}(v_{1i}, V_{2i}, b_{1i}) \mid B_{1,-i}^{\max} \leq b_{1i}, v_{1i}\right] \\ &= \mathbb{E}_{V_2|V_1}\left[\frac{H_2^{w}\left(\tilde{B}_{2i}^{w}; b_{1i}\right)^2}{h_2^{w}\left(\tilde{B}_{2i}^{w}; b_{1i}\right)} \mid v_{1i}\right], \end{aligned} \tag{3}$$

[6] Appendix A.5 includes an alternative derivation of $H_2^{w}(\cdot; b_{1i})$.

where $\tilde{B}^w_{2i} \equiv \tilde{s}^w_2\left(v_{1i}, \delta(v_{1i}, V_{2i}), b_{1i}\right)$ is bidder i's second auction optimal bid when he has won the first auction with a bid b_{1i}. I will use $\partial \mathcal{V}^w(v_{1i}, b_{1i})/\partial b_{1i}$ in (9).

$$\frac{\partial \mathcal{V}^w\left(v_{1i}, b_{1i}\right)}{\partial b_{1i}} = \frac{dG_1\left(b_{1i}\right)^{I-1}/db_{1i}}{G_1\left(b_{1i}\right)^{I-1}} \times \tag{4}$$
$$\mathbb{E}_{V_2|V_1}\left[\frac{H^w_2(\tilde{B}^w_{2i}; b_{1i})}{h^w_2(\tilde{B}^w_{2i}; b_{1i})}\left[G^l_2(\tilde{B}^w_{2i} \mid B_1 \leq b_{1i})^{I-2} G^l_{2|1}(\tilde{B}^w_{2i} \mid b_{1i}) - H^w_2(\tilde{B}^w_{2i}; b_{1i})\right] \mid v_{1i}\right].$$

When i is the First Auction Loser: Let the first auction winner be a bidder $k \neq i$. At the start of the second auction or *step*(iii), only thing a bidder i knows is $\{W_1 = k\} = \{B_{1k} > b_{1i}, B_{1k} > B_{1j}, j \notin \{i, k\}\}$. Given the condition the distribution of the highest competing bid $B^{\max}_{2,-i}$ in the second auction is,

$$\begin{aligned}
H^l_2\left(\cdot; b_{1i}\right) &\equiv \Pr\left[B^{\max}_{2,-i} \leq \cdot \mid B_{1k} > b_{1i}, B_{1k} > B_{1j}, j \notin \{i, k\}, V_{1i} = v_{1i}, V_{2i} = v_{2i}\right] \\
&= \Pr[s^w_2(V_{1k}, \delta(V_{1k}, V_{2k})) \leq \cdot, s^l_2(V_{1j}, V_{2j}) \leq \cdot, j \notin \{i, k\} \\
&\qquad \mid s_1(V_{1k}) > b_{1i}, s_1(V_{1k}) > s_1(V_{1j}), j \notin \{i, k\}, V_{1i} = v_{1i}, V_{2i} = v_{2i}] \\
&= \Pr\left[B^{\max}_{2,-i} \leq \cdot \mid B^{\max}_{1,-i} > b_{1i}\right] \\
&= \frac{1}{\Pr\left[B^{\max}_{1,-i} > b_{1i}\right]} \int_{b_{1i}}^{\overline{b_1}} \Pr\left[B^{\max}_{2,-i} \leq \cdot \mid B^{\max}_{1,-i} = x\right] d\Pr\left[B^{\max}_{1,-i} \leq x\right],
\end{aligned}$$

where the third row holds by Assumption 5. I have an equivalent expression for $H^l_2(\cdot; b_{1i})$ in (5) because $\Pr[B^{\max}_{2,-i} \leq \cdot \mid B^{\max}_{1,-i} = x] = G^l_2\left(\cdot \mid B_1 \leq x\right)^{I-2} G^w_{2|1}(\cdot \mid x)$ holds for $b_{1i} \leq x \leq \overline{b_1}$ as shown in Kong (2021).

$$H^l_2\left(\cdot; b_{1i}\right) = \frac{1}{1 - G_1\left(b_{1i}\right)^{I-1}} \int_{b_{1i}}^{\overline{b_1}} G^l_2\left(\cdot \mid B_1 \leq x\right)^{I-2} G^w_{2|1}(\cdot \mid x) dG_1(x)^{I-1}, \tag{5}$$

since $\Pr\left[B^{\max}_{1,-i} \leq x\right] = G_1(x)^{I-1}$. Using $H^l_2(\cdot; b_{1i})$ the expected profit from choosing b^l_{2i} in the second auction is $\pi^l_2\left(v_{2i}, b_{1i}, b^l_{2i}\right) \equiv \left(v_{2i} - b^l_{2i}\right) H^l_2\left(b^l_{2i}; b_{1i}\right)$ given (v_{2i}, b_{1i}) are fixed. The first-order condition of $\pi^l_2(v_{2i}, b_{1i}, b^l_{2i})$ with respect to b^l_{2i} is,

$$v_{2i} = b^l_{2i} + \frac{H^l_2\left(b^l_{2i}; b_{1i}\right)}{h^l_2\left(b^l_{2i}; b_{1i}\right)} \equiv \xi^l_2(b_{1i}, b^l_{2i}). \tag{6}$$

Assume that $\xi^l_2(b_{1i}, b^l_{2i})$ is increasing in b^l_{2i} so that by Lemma 2 I have a unique solution denoted as $\tilde{b}^l_{2i} \equiv \tilde{s}^l_2\left(v_{2i}, b_{1i}\right)$. Using $\tilde{b}^l_{2i}$ the expected profit in the second auction upon losing the first auction is,

$$\tilde{\pi}^l_2\left(v_{2i}, b_{1i}\right) = H^l_2\left(\tilde{b}^l_{2i}; b_{1i}\right)^2 / h^l_2\left(\tilde{b}^l_{2i}; b_{1i}\right).$$

A bidder i uses $\tilde{\pi}^l_2\left(v_{2i}, b_{1i}\right)$ at *step*(i). Assume that he has drawn v_{1i} and made a bid b_{1i} and is unaware of the outcome of the first auction. By the unawareness the continuation value for losing the first auction at *step*(i) relies on $\tilde{\pi}_2^l(v_{2i}, b_{1i})$.

$$\begin{aligned}\mathcal{V}^l(v_{1i}, b_{1i}) &\equiv \mathbb{E}_{V_2|V_1}\left[\tilde{\pi}_2^l(V_{2i}, b_{1i}) \mid B_{1,-i}^{\max} > b_{1i}, v_{1i}\right] \\ &= \mathbb{E}_{V_2|V_1}\left[\frac{H_2^l\left(\tilde{B}_{2i}^l; b_{1i}\right)^2}{h_2^l\left(\tilde{B}_{2i}^l; b_{1i}\right)} \mid v_{1i}\right],\end{aligned} \tag{7}$$

where $\tilde{B}_{2i}^l \equiv \tilde{s}_2^l(V_{2i}, b_{1i})$ is bidder i 's second-auction optimal bid when he has lost the first auction with a bid b_{1i}. I will use $\partial\mathcal{V}^l(v_{1i}, b_{1i})/\partial b_{1i}$ in (9).

$$\begin{aligned}&\frac{\partial\mathcal{V}^l(v_{1i}, b_{1i})}{\partial b_{1i}} = \frac{dG_1(b_{1i})^{I-1}/db_{1i}}{1 - G_1(b_{1i})^{I-1}}\times \\ &\mathbb{E}_{V_2|V_1}\left[\frac{H_2^l(\tilde{B}_{2i}^l; b_{1i})}{h_2^l(\tilde{B}_{2i}^l; b_{1i})}\left[H_2^l(\tilde{B}_{2i}^l; b_{1i}) - G_2^l(\tilde{B}_{2i}^l \mid B_1 \le b_{1i})^{I-2}G_{2|1}^w(\tilde{B}_{2i}^l \mid b_{1i})\right] \mid v_{1i}\right].\end{aligned} \tag{8}$$

The First and Second Auction strategies: A bidder i is at *step*(i). The expected profit from choosing b_{1i} in the first auction given v_{1i} is,

$$\pi(v_{1i}, b_{1i}) = \left[v_{1i} - b_{1i} + \mathcal{V}^w(v_{1i}, b_{1i})\right]G_1(b_{1i})^{I-1} + \mathcal{V}^l(v_{1i}, b_{1i})\left[1 - G_1(b_{1i})^{I-1}\right].$$

Differentiating $\pi(v_{1i}, b_{1i})$ with respect to b_{1i} gives the first-order condition,

$$\begin{aligned}v_{1i} &= b_{1i} + \frac{1}{I-1}\frac{G_1(b_{1i})}{g_1(b_{1i})} - \frac{\partial\left\{\mathcal{V}^w(v_{1i}, b_{1i})G_1(b_{1i})^{I-1} + \mathcal{V}^l(v_{1i}, b_{1i})[1 - G_1(b_{1i})^{I-1}]\right\}/\partial b_{1i}}{dG_1(b_{1i})^{I-1}/db_{1i}} \\ &= b_{1i} + \frac{1}{I-1}\frac{G_1(b_{1i})}{g_1(b_{1i})} - \mathcal{V}^w(v_{1i}, b_{1i}) + \mathcal{V}^l(v_{1i}, b_{1i}) \\ &\quad - \frac{\partial\mathcal{V}^w(v_{1i}, b_{1i})}{\partial b_{1i}}\frac{G_1(b_{1i})^{I-1}}{dG_1(b_{1i})^{I-1}/db_{1i}} - \frac{\partial\mathcal{V}^l(v_{1i}, b_{1i})}{\partial b_{1i}}\frac{[1 - G_1(b_{1i})^{I-1}]}{dG_1(b_{1i})^{I-1}/db_{1i}}.\end{aligned}$$

Using (3)-(4) and (7)-(8), the first-order condition is equivalent to the following,

$$\begin{aligned}v_{1i} &= b_{1i} + \frac{1}{I-1}\frac{G_1(b_{1i})}{g_1(b_{1i})} \\ &- \mathbb{E}_{V_2|V_1}\left[\frac{H_2^w(\tilde{B}_{2i}^w; b_{1i})}{h_2^w(\tilde{B}_{2i}^w; b_{1i})}G_2^l(\tilde{B}_{2i}^w \mid B_1 \le b_{1i})^{I-2}G_{2|1}^l(\tilde{B}_{2i}^w \mid b_{1i}) \mid v_{1i}\right] \\ &+ \mathbb{E}_{V_2|V_1}\left[\frac{H_2^l(\tilde{B}_{2i}^l; b_{1i})}{h_2^l(\tilde{B}_{2i}^l; b_{1i})}G_2^l(\tilde{B}_{2i}^l \mid B_1 \le b_{1i})^{I-2}G_{2|1}^w(\tilde{B}_{2i}^l \mid b_{1i}) \mid v_{1i}\right],\end{aligned} \tag{9}$$

where $\tilde{B}_{2i}^w \equiv \tilde{s}_2^w(v_{1i}, \delta(v_{1i}, V_{2i}), b_{1i})$ and $\tilde{B}_{2i}^l \equiv \tilde{s}_2^l(V_{2i}, b_{1i})$.

Assume the equilibrium where all the bidders including bidder i follow equilibrium strategies. The following holds for all the bidders at *step*(iii) given (v_{1i}, v_{2i}).

$$\begin{aligned}&b_{1i} = s_1(v_{1i}), b_{2i}^w = s_2^w(v_{1i}, \delta(v_{1i}, v_{2i})), b_{2i}^l = s_2^l(v_{1i}, v_{2i}), \\ &B_{1j} = s_1, B_{2j}^w = s_2^w, B_{2j}^l = s_2^l \quad \text{for } j \ne i.\end{aligned}$$

i's equilibrium bids $[b_{2i}^w, b_{2i}^l]$ must equal the optimal bids $[\tilde{b}_{2i}^w, \tilde{b}_{2i}^l]$ defined in (2) and (6). It implies that at *step*(iii) the following holds in equilibrium for bidder i with (v_{1i}, v_{2i}).

$$s_2^w(v_{1i}, \delta(v_{1i}, v_{2i})) = b_{2i}^w = \tilde{b}_{2i}^w \equiv \tilde{s}_2^w(v_{1i}, \delta(v_{1i}, v_{2i}), s_1(v_{1i})),$$
$$s_2^l(v_{1i}, v_{2i}) = b_{2i}^l = \tilde{b}_{2i}^l \equiv \tilde{s}_2^l(v_{2i}, s_1(v_{1i})).$$

The equivalences demonstrate the property that the strategies $[s_2^w(V_1, \cdot), s_2^l(V_1, \cdot)]$ are increasing for every V_1. $s_2^w(V_1, \cdot)$ exhibits increasing behavior by the following reasoning: $\xi_2^w(b_{1i}, b_{2i}^w)$ in (2) increases with b_{2i}^w for a given $b_{1i} = s_1(v_{1i})$ implying that an increase in b_{2i}^w leads to an overall increase in $\delta(v_{1i}, v_{2i})$. Given v_{1i} is fixed and $\delta(v_{1i}, \cdot)$ is increasing according to Assumption 4 it indicates that an increase in b_{2i}^w implies an increase in v_{2i}. By applying the same reasoning the decrease in b_{2i}^w implies a decrease in v_{2i}. Using contraposition the increase(decrease) in v_{2i} increases(decreases) b_{2i}^w. Since $b_{2i}^w = s_2^w(v_{1i}, \delta(v_{1i}, v_{2i}))$ holds in equilibrium, I prove the strategy s_2^w is increasing in v_{2i}. To demonstrate the increasing nature of $s_2^l(V_1, \cdot)$ I use the similar line of reasoning, substituting (6) and $b_{2i}^l = s_2^l(v_{1i}, v_{2i})$ for (2) and $b_{2i}^w = s_2^w(v_{1i}, \delta(v_{1i}, v_{2i}))$.

In equilibrium at *step*(i) the following hold for a bidder i with v_{1i}; V_{2i} is a random variable at the step.

$$b_{1i} = s_1(v_{1i}),$$
$$s_2^w(v_{1i}, \delta(v_{1i}, V_{2i})) = B_{2i}^w = \tilde{B}_{2i}^w \equiv \tilde{s}_2^w(v_{1i}, \delta(v_{1i}, V_{2i}), s_1(v_{1i})),$$
$$s_2^l(v_{1i}, V_{2i}) = B_{2i}^l = \tilde{B}_{2i}^l \equiv \tilde{s}_2^l(V_{2i}, s_1(v_{1i})),$$

where $\tilde{B}_{2i}^w$ and $\tilde{B}_{2i}^l$ were defined in (3) and (7). Using $B_{2i}^w = \tilde{B}_{2i}^w$ and $B_{2i}^l = \tilde{B}_{2i}^l$ I replace $\tilde{B}_{2i}^w$ with B_{2i}^w and $\tilde{B}_{2i}^l$ with B_{2i}^l in (9). The replacement yields each conditional expectation being taken over $B_2^w|V_1$ and $B_2^l|V_1$. If $s_1(\cdot)$ is increasing(which I will verify soon), conditioning on $V_1 = v_{1i}$ is equivalent to conditioning on $B_1 = b_{1i}$. It implies that the conditional expectations $B_2^w|V_1$ and $B_2^l|V_1$ equal $B_2^w|B_1 \sim G_{2|1}^w(\cdot|b_{1i})$ and $B_2^l|B_1 \sim G_{2|1}^l(\cdot|b_{1i})$. The equivalences transform (9) into (10).

$$\begin{aligned} v_{1i} = b_{1i} &+ \frac{1}{I-1}\frac{G_1(b_{1i})}{g_1(b_{1i})} \\ &- \mathbb{E}_{B_2^w|B_1}\left[\frac{H_2^w(B_{2i}^w; b_{1i})}{h_2^w(B_{2i}^w; b_{1i})} G_2^l(B_{2i}^w \mid B_1 \le b_{1i})^{I-2} G_{2|1}^l(B_{2i}^w \mid b_{1i}) \mid b_{1i}\right] \\ &+ \mathbb{E}_{B_2^l|B_1}\left[\frac{H_2^l(B_{2i}^l; b_{1i})}{h_2^l(B_{2i}^l; b_{1i})} G_2^l\left(B_{2i}^l \mid B_1 \le b_{1i}\right)^{I-2} G_{2|1}^w\left(B_{2i}^l \mid b_{1i}\right) \mid b_{1i}\right] \equiv \xi_1(b_{1i}). \end{aligned} \tag{10}$$

Assume that $\xi_1(b_{1i})$ is increasing in b_{1i} so that by Lemma 2 I have a unique solution denoted as $\tilde{b}_{1i}$, which implies $v_{1i} = \xi_1(\tilde{b}_{1i}) \Leftrightarrow \xi_1^{-1}(v_{1i}) = \tilde{b}_{1i}$. In equilibrium the optimal bid $\tilde{b}_{1i}$ must equal $b_{1i} = s_1(v_{1i})$ resulting in $\tilde{b}_{1i} = b_{1i} = s_1(v_{1i})$. Since $\xi_1^{-1}(v_{1i}) = \tilde{b}_{1i}$ I have $\xi_1^{-1}(v_{1i}) = \tilde{b}_{1i} = b_{1i} = s_1(v_{1i})$ implying $s_1 = \xi_1^{-1}$. Given the assumption of ξ_1 being increasing, I conclude that s_1 is also increasing.

Lemma 2 verifies why the restrictions imposed on the right-hand side of (2), (6), and (10) imply the uniqueness of optimal bids.

Lemma 2 *Assume ξ_2^w and ξ_2^l exist. If $\xi_2^w(b_{1i}, b_{2i}^w)$(resp., $\xi_2^l(b_{1i}, b_{2i}^l)$) is increasing in b_{2i}^w(resp., b_{2i}^l) for every b_{1i}, unique optimal bid that satisfies (2)(resp., (6)) exists. If ξ_1 is increasing in b_{1i}, unique optimal bid that satisfies (10) exists.*

$G_2^l(\cdot|B_1 \leq \cdot)$ and $G_{2|1}^w(\cdot|\cdot)$ are necessary conditions for the existence of $\xi_2^w(b_{1i}, b_{2i}^w)$ and $\xi_2^l(b_{1i}, b_{2i}^l)$. Cases exist where the necessary conditions may not hold. For example, if two goods are perfect complements, $G_2^l(\cdot|B_1 \leq \cdot)$ fails to exist because the first auction losers forgo the second object, anticipating that the first auction winner will bid highly in the second auction. If two goods are perfect substitutes, $G_{2|1}^w(\cdot|\cdot)$ does not exist because the first auction winner forgoes the second object, feeling that the first object is sufficient. Two examples illustrate that our model is not applicable to every bid distribution, as discussed in detail in Theorem 3. The Theorem, analogous to Theorem 1 in Guerre et al. (2000), verifies that the strategies $[s_1, s_2^w, s_2^l]$ are monotone Bayesian Nash Equilibrium strategies.

Theorem 3 *Assuming that the bid distributions are absolutely continuous and satisfy the assumptions in Lemma 2, I can identify the model primitives, $[F_1, F_{2|1}, \delta]$, using the approach introduced in Section 3. Given the identified model primitives, the triplet $[\xi_1,\ \xi_2^w,\ \xi_2^l]$ from [(10), (2) ,(6)] are the quasi-inverse of the Bayesian Nash Equilibrium strategies, i.e., $\xi_1(b_1) = s_1^{-1}(b_1) = v_1$, $\xi_2^w(b_1, b_2^w) = (s_2^w)^{-1}(b_2^w; b_1) = \delta(v_1, v_2)$, and $\xi_2^l(b_1, b_2^l) = (s_2^l)^{-1}(b_2^l; b_1) = v_2$.*

Theorem 3 requires that the triplet $[\xi_1(b_1),\ \xi_2^w(b_1, b_2^w),\ \xi_2^l(b_1, b_2^l)]$ are increasing and differentiable with respect to $[b_1,\ b_2^w$ for any $b_1,\ b_2^l$ for any $b_1]$. The requirement on the triplet, in conjunction with the Theorem, establishes the following properties of the equilibrium strategies:

- first auction equilibrium strategy, $s_1 = \xi_1^{-1}$, is increasing and differentiable with respect to $v_1 \in [\xi_1(\underline{b_1}), \xi_1(\overline{b_1})]$.
- first auction loser's second auction equilibrium strategy, $s_2^l = (\xi_2^l)^{-1}$, is increasing and differentiable with respect to $v_2 \in \left[\xi_2^l(b_1, \underline{b_2}), \xi_2^l(b_1, \overline{b_2})\right]$ given any $v_1 \in [\xi_1(\underline{b_1}), \xi_1(\overline{b_1})]$.
- first auction winner's second auction equilibrium strategy, $s_2^w = (\xi_2^w)^{-1}$, is increasing and differentiable with respect to $v_2 \in [\xi_2^l(b_1, \underline{b_2}),\ \xi_2^l(b_1, \overline{b_2})]$ given any $v_1 \in [\xi_1(\underline{b_1}), \xi_1(\overline{b_1})]$.

When deriving the equilibrium strategies in Theorem 3, I restricted the triplet to have specific properties. It implies that under the restrictions the Bayesian Nash equilibrium strategies, $[s_1,\ s_2^l\ s_2^w]$, are only allowed to be monotone[7] and differentiable. Along with the two properties, the equilibrium strategies are less demanding to compute in programs as it does not involve solving differential equations; I still need numerical integration on the observed bids, but an iteration of the optimization procedure is not required. Since the strategies consist of observed bids, I can plot the strategies $[s_1,\ s_2^l\ s_2^w]$ as shown in Section 5.

3 Identification

I consider two cases of the observations available to the analyst:

[7]'monotone' and 'monotonicity' refer to 'strictly monotone' and 'strict monotonicity unless otherwise stated.

- Case 1: The dataset only includes the bids of the winners $(B_{1\ell}^{\max}, B_{2\ell}^{\max})$ and their identities $(W_{1\ell}, W_{2\ell})$ in each of the two auctions for any $\ell \in \{1, \dots, L\}$.

- Case 2: The dataset shows all the bids and bidders' identities in each of the two auctions for any $\ell \in \{1, \dots, L\}$.

Case 2 requires more information compared to Case 1. Since Case 1 is more common in practice it is best to address it first. As discussed in 3.2 addressing Case 1 implies addressing Case 2.

I assume that the observed bids result from the equilibrium play throughout Section 3. The equilibrium play assumption implies that the bidders' bids satisfy the first-order conditions (2), (6), and (10). The three conditions connect bid distributions to bidders' valuations, and the connection relies on the assumptions stated in Lemma 2. I maintain the same assumptions when identifying the model primitives $[F_1, F_{2|1}, \delta]$; so, it is the restriction imposed on the bid distributions that allows us to identify the primitives, rather than the reliance on a prior parametric specification.

3.1 Case 1

The observations I[8] see are $(B_{1\ell}^{\max}, W_{1\ell}, B_{2\ell}^{\max}, W_{2\ell}, Z_\ell, \mathcal{I}_\ell)$ where $\ell = 1, \dots, L$. $\mathcal{I}_\ell$ is the number of bidders in ℓ-th auction pair and it remains the same across the first and second auction by Assumption 1. $B_{t\ell}^{\max}$ is the maximum bid among $\{B_{t1}, \dots, B_{t\mathcal{I}_\ell}\}$; $W_{t\ell}$ is the index of the random winner in the t-th auction within the ℓ-th auction pair; Z_ℓ is the observed characteristic in ℓ-th auction pair. It may consist of characteristics $Z_{1\ell}$ of the first auctioned object, characteristics $Z_{2\ell}$ of the second auctioned object, and interactions between $Z_{1\ell}$ and $Z_{2\ell}$.

For the rest of 3.1 assume that I am interested in identifying the model primitives given $(Z = z, \mathcal{I} = I)$ so that the primitives are $[F_1(\cdot|z, I),\ F_{2|1}(\cdot|\cdot, z, I),\ \delta(V_1, V_2; z)]$. To maintain brevity I will omit writing the $(Z = z, \mathcal{I} = I)$ in 3.1. Given the limited information available in Case 1, I require multiples steps 3.1.1-3.1.5 for identification.

3.1.1 Identification of $G_{2|1}^w(\cdot \mid \cdot)$ and $G_2^l(\cdot \mid B_1 \leq \cdot)$

Without loss of generality assume that a bidder i won the first auction with $b_1 \equiv b_{1i}$. From Lemma 1 I know that the second auction bids $(B_{2j}, j = 1 \dots, I)$ are independent with distributions $G_{2|1}^w(\cdot \mid b_1)$ for i and $G_2^l(\cdot \mid B_1 \leq b_1)$ for $j \neq i$, conditional on the event $\{B_{1,-i}^{\max} \leq B_{1i} = b_1\} = \{W_1 = i, B_1^{\max} = b_1\}$. Given the conditional independence of the second auction bids, I use Lemma 4 with $H_j(\cdot \mid b_1) = \Pr[B_2^{\max} \leq \cdot, W_2 = j \mid W_1 = i, B_1^{\max} = b_1]$ for $j = 1, \dots, I$.

[8] Throughout Sections 3 and 4, 'I' is equivalent to 'the analyst.'

For $j \neq i$ I can rewrite $H_j(\cdot \mid b_1)$ as (11).

$$\begin{aligned} H_j(\cdot \mid b_1) &= \frac{1}{I-1} \Pr\left[B_2^{\max} \le \cdot, W_2 \neq i \mid W_1 = i, B_1^{\max} = b_1\right] \qquad (11) \\ &= \frac{1}{I-1} \Pr\left[B_2^{\max} \le \cdot \mid W_2 \neq i, W_1 = i, B_1^{\max} = b_1\right] \times \Pr\left[W_2 \neq i \mid W_1 = i, B_1^{\max} = b_1\right] \\ &= \frac{1}{I-1} \Pr\left[B_2^{\max} \le \cdot \mid W_2 \neq W_1, W_1 = i, B_1^{\max} = b_1\right] \times \Pr\left[W_2 \neq W_1 \mid W_1 = i, B_1^{\max} = b_1\right] \\ &= \frac{1}{I-1} \Pr\left[B_2^{\max} \le \cdot \mid W_2 \neq W_1, B_1^{\max} = b_1\right] \times \Pr\left[W_2 \neq W_1 \mid B_1^{\max} = b_1\right] \\ &= \frac{1}{I-1} \Pr\left[B_2^{\max} \le \cdot, W_2 \neq W_1 \mid B_1^{\max} = b_1\right] \equiv \frac{1}{I-1} M_2^l(\cdot \mid b_1). \end{aligned}$$

When $j = i$ I have,

$$\begin{aligned} H_i(\cdot \mid b_1) &= \Pr\left[B_2^{\max} \le \cdot, W_2 = i \mid W_1 = i, B_1^{\max} = b_1\right] \qquad (12) \\ &= \Pr\left[B_2^{\max} \le \cdot \mid W_2 = i, W_1 = i, B_1^{\max} = b_1\right] \times \Pr\left[W_2 = i \mid W_1 = i, B_1^{\max} = b_1\right] \\ &= \Pr\left[B_2^{\max} \le \cdot \mid W_2 = W_1, W_1 = i, B_1^{\max} = b_1\right] \times \Pr\left[W_2 = W_1 \mid W_1 = i, B_1^{\max} = b_1\right] \\ &= \Pr\left[B_2^{\max} \le \cdot \mid W_2 = W_1, B_1^{\max} = b_1\right] \times \Pr\left[W_2 = W_1 \mid B_1^{\max} = b_1\right] \\ &= \Pr\left[B_2^{\max} \le \cdot, W_2 = W_1 \mid B_1^{\max} = b_1\right] \equiv M_2^w(\cdot \mid b_1). \end{aligned}$$

$M_2^l(\cdot \mid b_1)$ and $M_2^w(\cdot \mid b_1)$ in (11)-(12) are the distributions of the winning bid in the second auction when it is won by a first auction loser and first auction winner, conditional on the first auction winning bid being b_1[9]. Given (11)-(12) I use Lemma 4 to identify $G_{2|1}^w(\cdot \mid b_1)$ and $G_2^l(\cdot \mid B_1 \le b_1)$: $F_i(\cdot)$ and $F_j(\cdot)$ in Lemma 4 are $G_{2|1}^w(\cdot \mid b_1)$ and $G_2^l(\cdot \mid B_1 \le b_1)$.

$$G_{2|1}^w(\cdot \mid b_1) = \exp\left\{-\int_{\cdot}^{\overline{b_2}} \left(\Pr\left[B_2^{\max} \le b \mid B_1^{\max} = b_1\right]\right)^{-1} dM_2^w(b \mid b_1)\right\}, \qquad (13)$$

$$G_2^l(\cdot \mid B_1 \le b_1) = \exp\left\{-\frac{1}{I-1}\int_{\cdot}^{\overline{b_2}} \left(\Pr\left[B_2^{\max} \le b \mid B_1^{\max} = b_1\right]\right)^{-1} dM_2^l(b \mid b_1)\right\}, \qquad (14)$$

since $H_i(\cdot \mid b_1) + \sum_{j \neq i} H_j(\cdot \mid b_1) = M_2^w(\cdot \mid b_1) + M_2^l(\cdot \mid b_1) = \Pr\left[B_2^{\max} \le \cdot \mid B_1^{\max} = b_1\right]$. By varying b_1 I identify $G_{2|1}^w(\cdot \mid \cdot)$ and $G_2^l(\cdot \mid B_1 \le \cdot)$.

3.1.2 Identification of $H_2^w(\cdot;\cdot)$, $H_2^l(\cdot;\cdot)$, and $G_1(\cdot)$

From (1) and (5) in 2.2, I have

$$\begin{aligned} H_2^w(\cdot; b_1) &= G_2^l(\cdot \mid B_1 \le b_1)^{I-1}, \\ H_2^l(\cdot; b_1) &= \frac{1}{1 - G_1(b_1)^{I-1}} \int_{b_1}^{\overline{b_1}} G_2^l(\cdot \mid B_1 \le x)^{I-2} G_{2|1}^w(\cdot \mid x) dG_1(x)^{I-1}. \end{aligned}$$

I have identified $G_2^l(\cdot|B_1 \le \cdot)$ and $G_{2|1}^w(\cdot|\cdot)$ in 3.1.1. I can also identify $G_1(\cdot)$ by observing the first auction winning bids($B_1^{\max}$), which follows a distribution of $G_1(\cdot)^I$. It implies the following

[9] Strictly speaking, $M_2^l(\cdot \mid b_1)$ and $M_2^w(\cdot \mid b_1)$ are not distributions as they do not integrate to one.

equation.

$$G_1(\cdot) = \Pr[B_1^{\max} \leq \cdot]^{1/I}.$$

By varying b_1 I identify $H_2^w(\cdot;\cdot)$ and $H_2^l(\cdot;\cdot)$.

3.1.3 Identification of $\tilde{D}(\cdot \mid \cdot)$, $G_{2|1}^l(\cdot \mid \cdot)$, and $\tilde{F}_{2|1}(\cdot \mid \cdot)$

I introduce new notations that are comparable to the notations in 2.1.

$\tilde{\delta}(B_1, V_2)$: Consider a scenario where $B_1 = b_1$ is given. $\tilde{\delta}(B_1 = b_1, V_2 = x)$ represents the value of the second object when the first auction winner(w) possesses $B_1 = b_1$ worth of the first object.

$\tilde{D}(\cdot|b_1)$: $\Pr[\tilde{\delta}(b_1, V_2) \leq \cdot|B_1 = b_1]$; the distribution of the value of the second object for the first auction winner(w) given that he won the first auction with $B_1 = b_1$.

$\tilde{F}_{2|1}(\cdot|b_1)$: $\Pr[V_2 \leq \cdot|B_1 = b_1]$; the distribution of the value of the second object for the first auction loser(l) given that he lost the first auction with $B_1 = b_1$.

$\tilde{\delta}(B_1, V_2)$ is equivalent to $\delta(V_1, V_2)$, i.e., $\delta(V_1, V_2) = \tilde{\delta}(s_1(V_1), V_2)$ as shown in Appendix A.7. By the equivalence the first-order condition, (2), for an arbitrary bidder i is,

$$\tilde{\delta}\left(B_{1i}, V_{2i}\right) = B_{2i}^w + \frac{H_2^w\left(B_{2i}^w; B_{1i}\right)}{h_2^w\left(B_{2i}^w; B_{1i}\right)}, \tag{15}$$

where $B_{2i}^w = s_2^w\left(V_{1i}, V_{2i}\right)$ and $B_{1i} = s_1\left(V_{1i}\right)$. Note that the observations only include the winner's bid and identity, $(B_t^{\max}, W_t)$, for $t = 1, 2$. By the limitation, I observe (B_{1i}, B_{2i}^w) in (15) only if i wins both the first and second auctions. It implies that unless I condition on $W_1 = W_2 = i$ I cannot always recover $\tilde{\delta}\left(B_{1i}, V_{2i}\right)$ for every bidder i from (15).

To address the problem, I instead construct $\tilde{D}(\cdot|b_1)$ and $\tilde{F}_{2|1}(\cdot|b_1)$ in 3.1.3 and identify $\tilde{\delta}\left(b_1, \cdot\right)$ in 3.1.4. To construct $\tilde{D}(d|b_1)$ I use $G_{2|1}^w\left(\cdot \mid b_1\right)$.

$$\begin{aligned}
\tilde{D}(d|b_1) \equiv \Pr\left[\tilde{\delta}(B_1, V_2) \leq d \mid B_1 = b_1\right] &= \Pr\left[\underbrace{B_2^w + \frac{H_2^w(B_2^w; B_1)}{h_2^w(B_2^w; B_1)}}_{\equiv \xi_2^w(B_1, B_2^w) \text{ by } (2)} \leq d \mid B_1 = b_1\right] \qquad (16)\\
&= \mathbb{E}_{B_2^w|B_1}\left[\mathbb{1}\left(\xi_2^w(B_1, B_2^w) \leq d\right) \mid B_1 = b_1\right] \\
&= \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left(\xi_2^w(B_1, b) \leq d\right) dG_{2|1}^w(b|b_1),
\end{aligned}$$

where $d \in [\xi_2^w(b_1, \underline{b_2}), \xi_2^w(b_1, \overline{b_2})]$. Since I have identified $G_{2|1}^w\left(\cdot \mid b_1\right)$, $H_2^w\left(\cdot; b_1\right)$, and its density $h_2^w\left(\cdot; b_1\right)$ in 3.1.1-3.1.2 I conclude that $\tilde{D}\left(\cdot \mid b_1\right)$ is also identified. By varying b_1 I identify $\tilde{D}\left(\cdot \mid \cdot\right)$.

Before constructing $\tilde{F}_{2|1}(\cdot|b_1)$ I show that the observations I have prevent us from constructing $\tilde{F}_{2|1}(\cdot|b_1)$ directly from the first-order condition, (6). For an arbitrary bidder i I have the following.

$$V_{2i} = B_{2i}^l + \frac{H_2^l\left(B_{2i}^l; B_{1i}\right)}{h_2^l\left(B_{2i}^l; B_{1i}\right)}, \tag{17}$$

where $B_{2i}^l = s_2^l\left(V_{1i}, V_{2i}\right)$ and $B_{1i} = s_1\left(V_{1i}\right)$. Since winning bids are only shown, I see B_{2i}^l only if i loses the first auction and wins the second auction so that $B_2^{\max} = B_{2i}^l$. But if i loses the first auction I do not observe his bid B_{1i} as $B_1^{\max} \neq B_{1i}$. It implies that I cannot directly recover V_{2i} for every bidder i from (17).

To overcome the difficulty, I instead use $G_{2|1}^l\left(\cdot \mid b_1\right)$ to construct $\tilde{F}_{2|1}(\cdot|b_1)$. I can identify $G_{2|1}^l(\cdot \mid b_1)$ by using the relationship $G_2^l\left(\cdot \mid B_1 \leq b_1\right) = \left[1/G_1\left(b_1\right)\right] \int_{\underline{b_1}}^{b_1} G_{2|1}^l(\cdot \mid u) dG_1(u)$. Differentiating both sides of the equation with respect to b_1 yields (18),

$$G_{2|1}^l\left(\cdot \mid b_1\right) = G_2^l\left(\cdot \mid B_1 \leq b_1\right) + \frac{G_1\left(b_1\right)}{g_1\left(b_1\right)} \frac{\partial G_2^l\left(\cdot \mid B_1 \leq b_1\right)}{\partial b_1}. \tag{18}$$

The right-hand side of (18) has $G_2^l\left(\cdot \mid B_1 \leq b_1\right), G_1\left(b_1\right)$, and their derivatives which have been identified in 3.1.1-3.1.2. Using $G_{2|1}^l(\cdot \mid b_1)$ I have (19) for $\tilde{F}_{2|1}(\cdot \mid b_1)$.

$$\begin{aligned} \tilde{F}_{2|1}\left(v_2 \mid b_1\right) &\equiv \Pr\left[V_2 \leq v_2 \mid B_1 = b_1\right] = \Pr\left[\underbrace{B_2^l + \frac{H_2^l(B_2^l; B_1)}{h_2^l(B_2^l; B_1)}}_{\equiv \xi_2^l(B_1, B_2^l) \text{ by (6)}} \leq v_2 \mid B_1 = b_1\right] \\ &= \mathbb{E}_{B_2^l|B_1}\left[\mathbb{1}\left(\xi_2^l(B_1, B_2^l) \leq v_2\right) \mid B_1 = b_1\right] \\ &= \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left(\xi_2^l(B_1, b) \leq v_2\right) dG_{2|1}^l\left(b \mid b_1\right), \end{aligned} \tag{19}$$

where $v_2 \in [\xi_2^l(b_1, \underline{b_2}), \xi_2^l(b_1, \overline{b_2})]$. Since I have identified $G_{2|1}^l(\cdot|b_1)$ in (18) and $H_2^l\left(\cdot; b_1\right)$ along with its density $h_2^l\left(\cdot; b_1\right)$ in 3.1.2, I conclude that $\tilde{F}_{2|1}\left(\cdot \mid b_1\right)$ is identified. By varying b_1 I identify $\tilde{F}_{2|1}\left(\cdot \mid \cdot\right)$.

3.1.4 Identification of $\tilde{\delta}(\cdot, \cdot)$

I adopt the approach proposed in Kong (2021). The approach fixes a first auction bid at b_1 and compares the second object's value distribution between the first auction winner and the loser, $\tilde{D}\left(\cdot \mid b_1\right)$ and $\tilde{F}_{2|1}\left(\cdot \mid b_1\right)$. Comparing the quantiles between the two identifies $\tilde{\delta}(b_1, \cdot)$ given Assumption 4 holds.

Assumption 4 assumes $\delta(V_1, \cdot)$ is increasing in V_2 for every V_1. Since $\delta(V_1, \cdot) = \tilde{\delta}(s_1(V_1), \cdot)$ holds $\tilde{\delta}(B_1, \cdot)$ is also increasing in V_2 for every $B_1 = s_1(V_1)$. By the monotonicity of $\tilde{\delta}(B_1, \cdot)$ the α-quantile of the random variable $\tilde{\delta}\left(b_1, V_2\right)$[10] equals $\tilde{\delta}\left(b_1, v_2(\alpha|b_1)\right)$, where $v_2(\alpha|b_1)$ is the α-quantile of $\tilde{F}_{2|1}(\cdot|b_1) \equiv \Pr[V_2 \leq \cdot|B_1 = b_1]$. I am using the property that the quantile is invariant to monotone transformation.

[10] $\tilde{\delta}\left(b_1, V_2\right)$ is a function of a random variable V_2. The function here is $\tilde{\delta}(b_1, \cdot)$ where b_1 is a specified value.

Given the property, I have the following equality for any $\alpha \in [0,1]$,

$$\tilde{D}\left(\tilde{\delta}\left(b_1, v_2(\alpha|b_1)\right) \mid b_1\right) = \alpha = \tilde{F}_{2|1}\left(v_2(\alpha|b_1) \mid b_1\right).$$

Since $\tilde{D}(d|b_1)$ is increasing in d[11] I can define the inverse distribution function $\tilde{D}^{-1}(\cdot|b_1)$ such that the following relation holds,

$$\tilde{\delta}\left(b_1, v_2(\alpha|b_1)\right) = \tilde{D}^{-1}\left(\underbrace{\tilde{F}_{2|1}\left(v_2(\alpha|b_1)|b_1\right)}_{=\alpha} \middle| b_1\right). \tag{20}$$

I have identified $\tilde{D}(\cdot|b_1)$ and $\tilde{F}_{2|1}(\cdot|b_1)$ in 3.1.3, so by varying α I identify $\tilde{\delta}(b_1,\cdot)$; figure 1 graphically depicts (20).

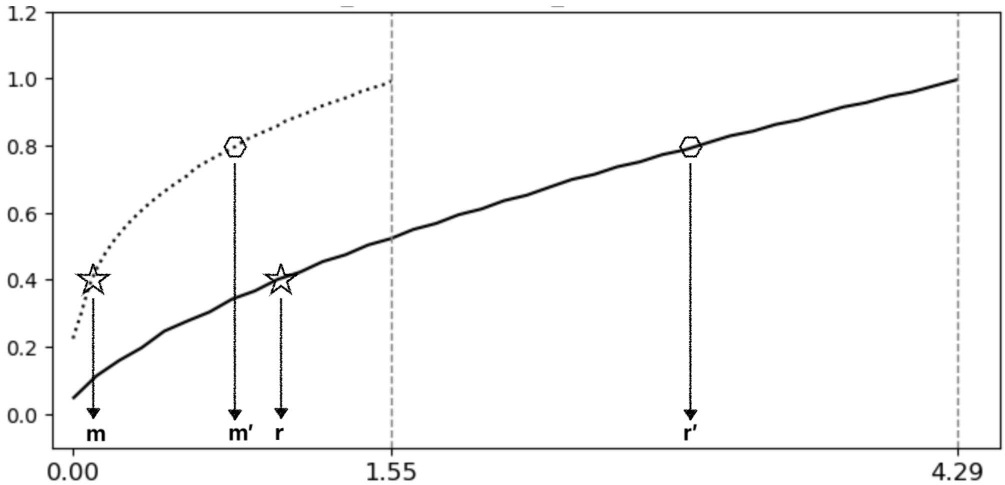

Figure 1: The solid line represents $\tilde{D}(d \mid b_1)$ from (16), with d varying between $[\xi_2^w(b_1,\underline{b_2}), \xi_2^w(b_1,\overline{b_2})] = [0,\ 4.29]$. The dotted line represents $\tilde{F}_{2|1}(v_2 \mid b_1)$ from (19), with v_2 varying between $[\xi_2^l(b_1,\underline{b_2}), \xi_2^l(b_1,\overline{b_2})] = [0,\ 1.55]$.

Set $\alpha = 0.4$, and let m and r represent the quantiles of the random variables V_2 and $\tilde{\delta}(b_1, V_2)$. Given m and r, if the pseudo synergy function $\tilde{\delta}(B_1,\cdot)$ is increasing in V_2 (Assumption 4), I have $\tilde{\delta}(b_1, v_2(0.4|b_1)) = \tilde{\delta}(b_1, m) = r$. If I change α to 0.8, I have $\tilde{\delta}(b_1, v_2(0.8|b_1)) = \tilde{\delta}(b_1, m') = r'$; another way to express the result is by using (20).

$$\tilde{\delta}\left(b_1, v_2\left(0.4|b_1\right)\right) = \tilde{\delta}(b_1, m) = r = \tilde{D}^{-1}(0.4 \mid b_1),$$
$$\tilde{\delta}\left(b_1, v_2\left(0.8|b_1\right)\right) = \tilde{\delta}(b_1, m') = r' = \tilde{D}^{-1}(0.8 \mid b_1).$$

It implies that by varying α between $[0,1]$ I identify a bijective function, $\tilde{\delta}(b_1, V_2) : [0.00, 1.55] \to [0.00, 4.29]$. Given $\tilde{\delta}(b_1,\cdot)$, I identify $\tilde{\delta}(\cdot,\cdot)$ by varying b_1.

[11] Consider the last equality in (16). Based on Lemma 2 and Theorem 3 the function $x + (H_2^w(x;b_1)/h_2^w(x;b_1))$ is increasing in x. It implies that as d increases the corresponding $\tilde{D}(d|b_1)$ also increases.

3.1.5 Identification of $F_1(\cdot), F_{2|1}(\cdot \mid \cdot)$ and $\delta(\cdot,\cdot)$

In 3.1.1-3.1.3 I have identified the distributions $G_1(\cdot)$, $H_2^w(\cdot;\cdot)$, $H_2^l(\cdot;\cdot)$, $G_{2|1}^w(\cdot \mid \cdot)$, $G_2^l(\cdot \mid B_1 \leq \cdot)$, $G_{2|1}^l(\cdot \mid \cdot)$ as well as the densities $g_1(\cdot)$, $h_2^w(\cdot;\cdot)$, $h_2^l(\cdot;\cdot)$. The distributions and the densities form the quasi-inverse bidding strategy $\xi_1(b_1)$ in (10) so $\xi_1(b_1)$ is identified. By varying b_1 I also identify $\xi_1(\cdot)$. (21) shows that identifying $\xi_1(\cdot)$ implies identifying $F_1(\cdot)$ since $G_1(\cdot)$ has been identified in 3.1.2 and $V_1 = \xi_1(B_1)$ holds from (10).

$$\begin{aligned} F_1(\cdot) &\equiv \Pr[V_1 \leq \cdot] = \Pr[\xi_1(B_1) \leq \cdot] = \mathbb{E}[\mathbb{1}(\xi_1(B_1) \leq \cdot)] \\ &= \int_{\underline{b_1}}^{\overline{b_1}} \mathbb{1}(\xi_1(b_1) \leq \cdot) dG_1(b_1). \end{aligned} \tag{21}$$

Identification of $F_{2|1}(\cdot \mid \cdot)$ uses Theorem 3. $\xi_1(\cdot) = s_1^{-1}(\cdot)$ holds which means that the first auction bidding strategy $s_1(\cdot)$ is identified. Given the increasing nature of $s_1(\cdot)$ I establish the following equivalence.

$$\begin{aligned} F_{2|1}(\cdot \mid v_1) &\equiv \Pr[V_2 \leq \cdot | V_1 = v_1] = \Pr[V_2 \leq \cdot | s_1(V_1) = s_1(v_1)] \\ &= \Pr[V_2 \leq \cdot | B_1 = b_1] \equiv \tilde{F}_{2|1}(\cdot \mid b_1), \end{aligned} \tag{22}$$

alternatively (22) is the same as $\tilde{F}_{2|1}(\cdot \mid b_1) \equiv \Pr[V_2 \leq \cdot | B_1 = b_1] = \Pr[V_2 \leq \cdot | \xi_1(B_1) = \xi_1(b_1)] = \Pr[V_2 \leq \cdot \mid V_1 = v_1] \equiv F_{2|1}(\cdot \mid v_1)$. The equivalence demonstrates that identifying $\tilde{F}_{2|1}(\cdot \mid b_1)$ is equivalent to identifying $F_{2|1}(\cdot \mid v_1)$. Since I have identified $\tilde{F}_{2|1}(\cdot \mid b_1)$ for every $b_1 \in [\underline{b_1}, \overline{b_1}]$ in 3.1.3 I conclude that I also identify $F_{2|1}(\cdot \mid v_1)$ for every $v_1 \in [\underline{V_1}, \overline{V_1}] = [\xi_1(\underline{b_1}), \xi_1(\overline{b_1})]$.

Identification of $\delta(\cdot,\cdot)$ uses the equality $\tilde{\delta}(b_1,\cdot) = \tilde{\delta}(s_1(v_1),\cdot) = \delta(v_1,\cdot)$. From the equality I conclude that identifying $\tilde{\delta}(b_1,\cdot)$ is equivalent to identifying $\delta(v_1,\cdot)$. Since $\tilde{\delta}(b_1,\cdot)$ was identified for every $b_1 \in [\underline{b_1}, \overline{b_1}]$ in 3.1.4 and $s_1(\cdot) = \xi_1^{-1}(\cdot)$ was identified, I conclude that I also identify $\delta(v_1,\cdot)$ for every $v_1 \in [\underline{V_1}, \overline{V_1}] = [\xi_1(\underline{b_1}), \xi_1(\overline{b_1})]$.

3.2 Case 2

I directly identify the second auction bid distributions of the first auction winner and the loser, $G_{2|1}^w(\cdot \mid \cdot)$ and $G_{2|1}^l(\cdot \mid \cdot)$. I also identify $G_2^l(\cdot \mid B_1 \leq \cdot)$ and $G_1(\cdot)$ from the observations. Using (1) and (5) the distributions $H_2^w(\cdot;\cdot)$ and $H_2^l(\cdot;\cdot)$ are identified. It implies that the identification tasks outlined in 3.1.1-3.1.2 are easily addressed. The remaining steps, 3.1.3-3.1.5, follow the same procedure as in Case 1.

4 Estimation

Assume that I am in Case 1(3.1) so that the observations I see are ($B_{1\ell}^{\max}$, $W_{1\ell}$, $B_{2\ell}^{\max}$, $W_{2\ell}$, Z_ℓ, $\mathcal{I}_\ell$) where $\ell = 1, \ldots, L$. Let the observed characteristic, Z_ℓ, be continuous and without loss of generality be of dimension $p = 1$. I fix $(Z = z, \mathcal{I} = I)$ and use the kernel density estimator to estimate the model primitives, $[F_1(\cdot|z, I)$, $F_{2|1}(\cdot|\cdot, z, I)$, $\delta(V_1, V_2; z)]$.

Given ℓ-th auction pair I can calculate $\lambda_\ell(b_1)$ and $\overline{K}_{2\ell}(b_2)$ for any b_1 and b_2.

$$\lambda_\ell(b_1) \equiv K\left(\frac{b_1 - B^{\max}_{1\ell}}{h_1}\right) K\left(\frac{z - Z_\ell}{h_z}\right) / \left[\sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B^{\max}_{1\ell}}{h_1}\right) K\left(\frac{z - Z_\ell}{h_z}\right)\right],$$

$$\overline{K}_{2\ell}(b_2) \equiv \int_{-\infty}^{\frac{b_2 - B^{max}_{2\ell}}{h_2}} K(u)du = \int_{-\infty}^{b_2} \frac{1}{h_2} K\left(\frac{x - B^{max}_{2\ell}}{h_2}\right) dx,$$

where $\mathcal{L}_I \equiv \{\ell : \mathcal{I}_\ell = I\}$ is the index set corresponding to auction pairs with I bidders. I use $\lambda_\ell(b_1)$ and $\overline{K}_{2\ell}(b_2)$ in (23)-(26); the equations use Assumption 2 that the auction pairs are independent across $\{1, \ldots, L\}$.

$$\begin{aligned} \hat{M}^w_2(b_2 \mid b_1, z, I) &\equiv \hat{\Pr}\left[B^{\max}_2 \le b_2, W_2 = W_1 \mid B^{\max}_1 = b_1, z, I\right] \\ &= \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \lambda_\ell(b_1) \underbrace{\int_{-\infty}^{b_2} \frac{1}{h_2} K\left(\frac{x - B^{max}_{2\ell}}{h_2}\right) dx}_{\overline{K}_{2\ell}(b_2)}, \end{aligned} \tag{23}$$

$$\hat{m}^w_2(b_2 | b_1, z, I) = \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \lambda_\ell(b_1) \frac{1}{h_2} K\left(\frac{b_2 - B^{max}_{2\ell}}{h_2}\right), \tag{24}$$

$$\begin{aligned} \hat{M}^l_2(b_2 \mid b_1, z, I) &\equiv \hat{\Pr}\left[B^{\max}_2 \le b_2, W_2 \ne W_1 \mid B^{\max}_1 = b_1, z, I\right] \\ &= \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \ne W_{2\ell}\}} \lambda_\ell(b_1) \underbrace{\int_{-\infty}^{b_2} \frac{1}{h_2} K\left(\frac{x - B^{max}_{2\ell}}{h_2}\right) dx}_{\overline{K}_{2\ell}(b_2)}, \end{aligned} \tag{25}$$

$$\hat{m}^l_2(b_2 | b_1, z, I) = \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \ne W_{2\ell}\}} \lambda_\ell(b_1) \frac{1}{h_2} K\left(\frac{b_2 - B^{max}_{2\ell}}{h_2}\right). \tag{26}$$

(23) and (25) constitute (27), the conditional distribution of the second auction's winning bid given the winning bid of the first auction.

$$\begin{aligned} \hat{G}_{B^{\max}_2 | B^{\max}_1}(b_2 \mid b_1, z, I) &\equiv \hat{\Pr}[B^{\max}_2 \le b_2 \mid B^{\max}_1 = b_1, z, I] \\ &= \hat{M}^w_2(b_2 \mid b_1, z, I) + \hat{M}^l_2(b_2 \mid b_1, z, I) = \sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1) \overline{K}_{2\ell}(b_2). \end{aligned} \tag{27}$$

(23)-(27) are used in 4.1-4.5. The estimands in each subsection correspond to those discussed in 3.1.1-3.1.5.

4.1 Estimation of $G^w_{2|1}(\cdot \mid \cdot, z, I)$, $G^l_2(\cdot \mid B_1 \le \cdot, z, I)$ and their densities

I construct $\hat{G}^w_{2|1}\left(b_2 \mid b_1, z, I\right)$ and $\hat{G}^l_2\left(b_2 \mid B_1 \le b_1, z, I\right)$ based on (13) and (14).

$$\hat{G}^w_{2|1}\left(b_2 \mid b_1, z, I\right) = \prod_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \exp\left\{ -\frac{\lambda_\ell(b_1)}{h_2} \int_{b_2}^{\overline{b_2}} \frac{K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell} \in \mathcal{L}_I} \lambda_{\tilde{\ell}}(b_1) \overline{K}_{2\tilde{\ell}}(b)} db \right\},$$

$$\hat{G}^l_2\left(b_2 \mid B_1 \le b_1, z, I\right) = \prod_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \exp\left\{ -\frac{\lambda_\ell(b_1)}{h_2(I-1)} \int_{b_2}^{\overline{b_2}} \frac{K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell} \in \mathcal{L}_I} \lambda_{\tilde{\ell}}(b_1) \overline{K}_{2\tilde{\ell}}(b)} db \right\}.$$

To construct $\hat{g}^w_{2|1}\left(b_2 \mid b_1, z, I\right)$ and $\hat{g}^l_2\left(b_2 \mid B_1 \le b_1, z, I\right)$ I differentiate (13) and (14) with respect to the second auction bid and replace variables with estimators.

$$\hat{g}^w_{2|1}\left(b_2 \mid b_1, z, I\right) = \frac{\hat{m}^w_2\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B^{\max}_2 | B^{\max}_1}\left(b_2 \mid b_1, z, I\right)} \hat{G}^w_{2|1}\left(b_2 \mid b_1, z, I\right),$$
$$\hat{g}^l_2\left(b_2 \mid B_1 \le b_1, z, I\right) = \frac{1}{I-1} \frac{\hat{m}^l_2\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B^{\max}_2 | B^{\max}_1}\left(b_2 \mid b_1, z, I\right)} \hat{G}^l_2\left(b_2 \mid B_1 \le b_1, z, I\right).$$

The estimators on the right-hand sides are already known.

4.2 Estimation of $H^w_2(\cdot; \cdot, z, I)$, $H^l_2(\cdot; \cdot, z, I)$, and $G_1(\cdot \mid z, I)$ and their densities

Given ℓ-th auction pair I can calculate $\overline{K}_{1\ell}(b_1)$ for any b_1, and ω_ℓ.

$$\overline{K}_{1\ell}(b_1) \equiv \int_{-\infty}^{\frac{b_1 - B^{max}_{1\ell}}{h_1}} K(u) du = \int_{-\infty}^{b_1} \frac{1}{h_1} K\left(\frac{x - B^{max}_{1\ell}}{h_1}\right) dx,$$
$$\omega_\ell \equiv K\left(\frac{z - Z_\ell}{h_z}\right) / \sum_{\ell \in \mathcal{L}_I} K\left(\frac{z - Z_\ell}{h_z}\right).$$

$\overline{K}_{1\ell}(b_1)$ and ω_ℓ are used in $\hat{G}_1(b_1 \mid z, I)$ and $\hat{g}_1(b_1 \mid z, I)$,

$$\hat{G}_1(b_1 \mid z, I) = \hat{\Pr}[B^{\max}_1 \le b_1 \mid z, I]^{1/I} = \left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(b_1)\right)^{1/I},$$

$$\hat{g}_1(b_1 \mid z, I) = \frac{1}{I} \left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}\left(b_1\right)\right)^{(1-I)/I} \left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \frac{1}{h_1} K\left(\frac{b_1 - B^{max}_{1\ell}}{h_1}\right)\right).$$

The estimators of $H_2^w(b_2; b_1, z, I)$ and $H_2^l(b_2; b_1, z, I)$ use (1) and (5) in 2.2,

$$\begin{aligned}
\hat{H}_2^w\left(b_2; b_1, z, I\right) &= \hat{G}_2^l\left(b_2 \mid B_1 \leq b_1, z, I\right)^{I-1}, \\
\hat{H}_2^l\left(b_2; b_1, z, I\right) &= \frac{1}{1-\hat{G}_1\left(b_1 \mid z, I\right)^{I-1}} \times \\
&\int_{b_1}^{\overline{b_1}} \hat{G}_2^l\left(b_2 \mid B_1 \leq x, z, I\right)^{I-2} \hat{G}_{2|1}^w(b_2 \mid x, z, I) d\hat{G}_1(x \mid z, I)^{I-1}.
\end{aligned}$$

From 4.1-4.2 I know the right-hand sides except $d\hat{G}_1(x \mid z, I)^{I-1} = \frac{d}{dx}\hat{G}_1(x \mid z, I)^{I-1}\, dx$, which is,

$$\frac{d}{dx}\hat{G}_1(x \mid z, I)^{I-1}\, dx = \frac{I-1}{h_1 I}\left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(x)\right)^{-1/I} \sum_{\ell \in \mathcal{L}_I} \omega_\ell K\left(\frac{x - B_{1\ell}^{max}}{h_1}\right) dx.$$

To construct $\hat{h}_2^w(b_2; b_1, z, I)$ and $\hat{h}_2^l(b_2; b_1, z, I)$ I differentiate (1) and (5) with respect to the second auction bid and replace variables with estimators.

$$\begin{aligned}
\hat{h}_2^w\left(b_2; b_1, z, I\right) &= (I-1)\hat{g}_2^l\left(b_2 \mid B_1 \leq b_1, z, I\right)\hat{G}_2^l\left(b_2 \mid B_1 \leq b_1, z, I\right)^{I-2}, \\
\hat{h}_2^l\left(b_2; b_1, z, I\right) &= \frac{1}{1-\hat{G}_1\left(b_1 \mid z, I\right)^{I-1}} \times \\
&\int_{b_1}^{\overline{b_1}} \hat{\Psi}(b_2; x, z, I)\hat{G}_2^l\left(b_2 \mid B_1 \leq x, z, I\right)^{I-2} \hat{G}_{2|1}^w(b_2 \mid x, z, I) d\hat{G}_1(x \mid z, I)^{I-1},
\end{aligned}$$

where $\hat{\Psi}(b_2; x, z, I)$ inside the integral is,

$$\begin{aligned}
\hat{\Psi}(b_2; x, z, I) &\equiv (I-2)\frac{\hat{g}_2^l\left(b_2 \mid B_1 \leq x, z, I\right)}{\hat{G}_2^l\left(b_2 \mid B_1 \leq x, z, I\right)} + \frac{\hat{g}_{2|1}^w(b_2 \mid x, z, I)}{\hat{G}_{2|1}^w(b_2 \mid x, z, I)} \\
&= \frac{I-2}{I-1}\frac{\hat{m}_2^l\left(b_2 \mid x, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid x, z, I\right)} + \frac{\hat{m}_2^w\left(b_2 \mid x, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid x, z, I\right)}.
\end{aligned}$$

The estimators that form $\hat{h}_2^w(b_2; b_1, z, I)$ and $\hat{h}_2^l(b_2; b_1, z, I)$ are known from 4.1-4.2.

4.3 Estimation of $\tilde{D}(\cdot \mid \cdot, z, I)$, $G_{2|1}^l(\cdot \mid \cdot, z, I)$, and $\tilde{F}_{2|1}(\cdot \mid \cdot, z, I)$

I have a plug-in estimator of $\tilde{D}\left(d \mid b_1, z, I\right)$ using (16).

$$\hat{\tilde{D}}\left(d \mid b_1, z, I\right) = \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left(x + \frac{\hat{H}_2^w\left(x; b_1, z, I\right)}{\hat{h}_2^w\left(x; b_1, z, I\right)} \leq d\right) d\hat{G}_{2|1}^w\left(x \mid b_1, z, I\right), \tag{28}$$

where $x + \hat{H}_2^w(x; b_1, z, I)/\hat{h}_2^w(x; b_1, z, I)$ is,

$$\hat{\xi}_2^w(b_1, x; z, I) \equiv x + \frac{\hat{H}_2^w\left(x; b_1, z, I\right)}{\hat{h}_2^w\left(x; b_1, z, I\right)} = x + \frac{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(x \mid b_1, z, I\right)}{\hat{m}_2^l\left(x \mid b_1, z, I\right)}. \tag{29}$$

(26) and (27) constitute $\hat{\xi}_2^w(b_1, x; z, I)$, which is the empirical analogue of $\xi_2^w(b_1, x; z, I)$ from (2). Since $\xi_2^w(b_1, x; z, I)$ is increasing in x by Theorem 3, I can conclude that for a given d in $\tilde{D}(d \mid b_1, z, I)$ there exists a unique second auction bid $b_2^{w*}(d)$ that satisfies $\xi_2^w(b_1, b_2^{w*}(d); z, I) = d$, i.e.,

$$\xi_2^w(b_1, b_2^{w*}(d); z, I) \equiv b_2^{w*}(d) + \frac{H_2^w\left(b_2^{w*}(d); b_1, z, I\right)}{h_2^w\left(b_2^{w*}(d); b_1, z, I\right)} = d,$$

where $d \in \left[\xi_2^w(b_1, \underline{b_2}; z, I),\, \xi_2^w(b_1, \overline{b_2}; z, I)\right]$. It implies that given any d in the specified range I have $\tilde{D}(d \mid b_1, z, I) = G_{2|1}^w(b_2^{w*}(d) \mid b_1, z, I)$. The approach, however, does not hold for $\hat{\tilde{D}}\left(d \mid b_1, z, I\right)$ because the empirical counterpart, $\hat{\xi}_2^w(b_1, x; z, I)$, may not be increasing, leading to a non-unique $\hat{b}_2^{w*}(d)$. To ensure uniqueness I define $\hat{b}_2^{w*}(d)$ as the minimizer of the following function.

$$\hat{b}_2^{w*}(d) \equiv \operatorname{argmin}_x \left(\hat{\xi}_2^w(b_1, x; z, I) - d\right)^2 \equiv \operatorname{argmin}_x \left(x + \frac{\hat{H}_2^w\left(x; b_1, z, I\right)}{\hat{h}_2^w\left(x; b_1, z, I\right)} - d\right)^2.$$

Since $\hat{b}_2^{w*}(d)$ is unique given some d, I transform (28) into (30); $\hat{G}_{2|1}^w(\cdot \mid b_1, z, I)$ inside (30) is known from 4.1.

$$\hat{\tilde{D}}(d \mid b_1, z, I) = \hat{G}_{2|1}^w(\hat{b}_2^{w*}(d) \mid b_1, z, I). \tag{30}$$

I have a plug-in estimator of $\tilde{F}_{2|1}(v_2 \mid b_1, z, I)$ using (19).

$$\hat{\tilde{F}}_{2|1}\left(v_2 \mid b_1, z, I\right) = \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left(x + \frac{\hat{H}_2^l\left(x; b_1, z, I\right)}{\hat{h}_2^l\left(x; b_1, z, I\right)} \leq v_2\right) d\hat{G}_{2|1}^l\left(x \mid b_1, z, I\right), \tag{31}$$

where $x + \hat{H}_2^l(x; b_1, z, I)/\hat{h}_2^l(x; b_1, z, I)$ is,

$$\begin{aligned} \hat{\xi}_2^l(b_1, x; z, I) &\equiv x + \frac{\hat{H}_2^l(x; b_1, z, I)}{\hat{h}_2^l(x; b_1, z, I)} \\ &= x + \frac{\int_{b_1}^{\overline{b_1}} \hat{G}_2^l\left(x \mid B_1 \leq b, z, I\right)^{I-2} \hat{G}_{2|1}^w(x \mid b, z, I) d\hat{G}_1(b \mid z, I)^{I-1}}{\int_{b_1}^{\overline{b_1}} \hat{\Psi}(x; b, z, I) \hat{G}_2^l\left(x \mid B_1 \leq b, z, I\right)^{I-2} \hat{G}_{2|1}^w(x \mid b, z, I) d\hat{G}_1(b \mid z, I)^{I-1}}. \end{aligned} \tag{32}$$

The estimators in 4.1-4.2 make up $\hat{\xi}_2^l(b_1, x; z, I)$, which is the empirical analogue of $\xi_2^l(b_1, x; z, I)$ from (6). $\xi_2^l(b_1, x; z, I)$ is increasing in x by Theorem 3 but $\hat{\xi}_2^l(b_1, x; z, I)$ may not. I use the same approach used in $\hat{\tilde{D}}(d \mid b_1, z, I)$, leading to,

$$\hat{b}_2^{l*}(v_2) \equiv \operatorname{argmin}_x \left(\hat{\xi}_2^l(b_1, x; z, I) - v_2\right)^2 \equiv \operatorname{argmin}_x \left(x + \frac{\hat{H}_2^l\left(x; b_1, z, I\right)}{\hat{h}_2^l\left(x; b_1, z, I\right)} - v_2\right)^2,$$

where $v_2 \in \left[\hat{\xi}_2^l(b_1, \underline{b_2}; z, I),\, \hat{\xi}_2^l(b_1, \overline{b_2}; z, I)\right]$. Since $\hat{b}_2^{l*}(v_2)$ is unique given some v_2, I transform

(31) into (33).

$$\hat{\hat{F}}_{2|1}(v_2 \mid b_1, z, I) = \hat{G}^l_{2|1}(\hat{b}^{l*}_2(v_2) \mid b_1, z, I). \tag{33}$$

It needs $\hat{G}^l_{2|1}(b_2 \mid b_1, z, I)$, which I haven't constructed yet. A plug-in estimator of $G^l_{2|1}(b_2 \mid b_1, z, I)$ uses (18).

$$\hat{G}^l_{2|1}(b_2 \mid b_1, z, I) = \hat{G}^l_2(b_2 \mid B_1 \le b_1, z, I) + \frac{\hat{G}_1(b_1 \mid z, I)}{\hat{g}_1(b_1 \mid z, I)} \frac{\partial \hat{G}^l_2(b_2 \mid B_1 \le b_1, z, I)}{\partial b_1}.$$

From 4.1-4.2 I know the right-hand side except $\partial \hat{G}^l_2(b_2 \mid B_1 \le b_1, z, I)/\partial b_1$, which is,

$$\frac{\partial \hat{G}^l_2(b_2 \mid B_1 \le b_1, z, I)}{\partial b_1} =$$

$$\frac{\hat{G}^l_2(b_2 \mid B_1 \le b_1, z, I)}{h_2(I-1)} \int_{\underline{b_2}}^{\overline{b_2}} \frac{\sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \lambda_\ell(b_1) K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)\left(\sum_{\ell \in \mathcal{L}_I} \frac{\partial \lambda_\ell(b_1)}{\partial b_1} \overline{K}_{2\ell}(b)\right)}{(\sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1) \overline{K}_{2\ell}(b))^2} db$$

$$- \frac{\hat{G}^l_2(b_2 \mid B_1 \le b_1, z, I)}{h_2(I-1)} \int_{\underline{b_2}}^{\overline{b_2}} \frac{\sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \frac{\partial \lambda_\ell(b_1)}{\partial b_1} K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1) \overline{K}_{2\ell}(b)} db,$$

where $\partial \lambda_\ell(b_1)/\partial b_1$ is as follows; $k(\cdot)$ is the derivative of $K(\cdot)$.

$$\frac{\partial \lambda_\ell(b_1)}{\partial b_1} = \frac{\lambda_\ell(b_1)}{h_1} \left(\frac{k\left(\frac{b_1 - B^{max}_{1\ell}}{h_1}\right)}{K\left(\frac{b_1 - B^{max}_{1\ell}}{h_1}\right)} - \frac{\sum_{\ell \in \mathcal{L}_I} k\left(\frac{b_1 - B^{max}_{1\ell}}{h_1}\right) K\left(\frac{z - Z_\ell}{h_z}\right)}{\sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B^{max}_{1\ell}}{h_1}\right) K\left(\frac{z - Z_\ell}{h_z}\right)} \right).$$

I already know the estimators that constitute the right-hand side of $\partial \hat{G}^l_2(b_2 \mid B_1 \le b_1, z, I)/\partial b_1$.

4.4 Estimation of $\tilde{\delta}(\cdot, \cdot; z)$

The identification strategy used in 3.1.4 applies here; I have a plug-in estimator of (20).

$$\hat{\hat{\delta}}(b_1, v_2(\alpha | b_1, z, I); z, I) = \hat{\hat{D}}^{-1}\left(\hat{\hat{F}}_{2|1}(\hat{v}_2(\alpha | b_1, z, I) | b_1, z, I) \,\Big|\, b_1, z, I\right), \tag{34}$$

where I use $\hat{\hat{F}}_{2|1}(\cdot \mid b_1, z, I)$ and $\hat{\hat{D}}(\cdot \mid b_1, z, I)$ from 4.3, and $\hat{v}_2(\alpha | b_1, z, I)$ represents the α-quantile of $\hat{\hat{F}}_{2|1}(\cdot \mid b_1, z, I)$. By varying $\alpha \in [0, 1]$ in (34) I obtain $\hat{\hat{\delta}}(b_1, \cdot; z, I)$. Let L_I be the number of auction pairs with I bidders, and then I define a new estimator (35) from (34).

$$\hat{\hat{\delta}}(b_1, \cdot; z) \equiv (\sum_{\tilde{I}=2}^{N} L_{\tilde{I}})^{-1} \sum_{I=2}^{N} L_I \hat{\hat{\delta}}(b_1, \cdot; z, I), \tag{35}$$

where I assumed that the maximum number of bidders possible is N. (35) weights $\hat{\hat{\delta}}(b_1, \cdot; z, I)$ by L_I, which implies that I need to have prior knowledge of $\hat{\hat{\delta}}(b_1, \cdot; z, I)$ for every $I \in \{2, \ldots, N\}$.

4.5 Estimation of $F_1(\cdot \mid z, I), F_{2|1}(\cdot \mid \cdot, z, I)$ and $\delta(\cdot, \cdot; z)$

A plug-in estimator of $F_1(\cdot \mid z, I)$ or (21) is,

$$\hat{F}_1(v_1 \mid z, I) = \int_{\underline{b_1}}^{\overline{b_1}} \mathbb{1}\left(\hat{\xi}_1(b_1; z, I) \leq v_1\right) d\hat{G}_1(b_1 \mid z, I),$$

where $v_1 \in [\hat{\xi}_1(\underline{b_1}; z, I), \hat{\xi}_1(\overline{b_1}; z, I)]$. I know $d\hat{G}_1(b_1 \mid z, I) = \hat{g}_1(b_1 \mid z, I)db_1$ from 4.2, but haven't constructed $\hat{\xi}_1(b_1; z, I)$ yet. A plug-in estimator of $\xi_1(b_1; z, I)$ or (10) is as follows; to maintain brevity I will omit writing $(Z = z, \mathcal{I} = I)$ as much as possible in 4.5.

$$\begin{aligned}
\hat{\xi}_1(b_1) \equiv & b_1 + \frac{1}{I-1}\frac{\hat{G}_1(b_1)}{\hat{g}_1(b_1)} \\
& - \int_{\underline{b_2}}^{\overline{b_2}} \left[\frac{\hat{H}_2^w(b_2; b_1)}{\hat{h}_2^w(b_2; b_1)} \hat{G}_2^l(b_2 \mid B_1 \leq b_1)^{I-2} \hat{G}_{2|1}^l(b_2 \mid b_1)\right] d\hat{G}_{2|1}^w(b_2|b_1) \\
& + \int_{\underline{b_2}}^{\overline{b_2}} \left[\frac{\hat{H}_2^l(b_2; b_1)}{\hat{h}_2^l(b_2; b_1)} \hat{G}_2^l(b_2 \mid B_1 \leq b_1)^{I-2} \hat{G}_{2|1}^w(b_2 \mid b_1)\right] d\hat{G}_{2|1}^l(b_2|b_1).
\end{aligned}$$

I know all the estimators on the right-hand side from 4.1-4.4 except $d\hat{G}_{2|1}^l(b_2 \mid b_1) = \hat{g}_{2|1}^l(b_2 \mid b_1)db_2$. Since $\hat{G}_{2|1}^l(b_2 \mid b_1)$ is given in 4.3, I differentiate the estimator with respect to b_2 to obtain $\hat{g}_{2|1}^l(b_2 \mid b_1)$.

$$\begin{aligned}
\hat{g}_{2|1}^l(b_2 \mid b_1) &= \frac{d}{db_2}\hat{G}_{2|1}^l(b_2 \mid b_1) \\
&= \hat{g}_2^l(b_2 \mid B_1 \leq b_1) + \frac{\hat{G}_1(b_1)}{\hat{g}_1(b_1)}\frac{\partial \hat{g}_2^l(b_2 \mid B_1 \leq b_1)}{\partial b_1} \\
&= \frac{1}{I-1}\frac{\hat{m}_2^l(b_2 \mid b_1)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1)}\hat{G}_2^l(b_2 \mid B_1 \leq b_1) \\
&\quad + \frac{1}{I-1}\frac{\hat{G}_1(b_1)}{\hat{g}_1(b_1)}\frac{\partial}{\partial b_1}\left(\frac{\hat{m}_2^l(b_2 \mid b_1)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1)}\right)\hat{G}_2^l(b_2 \mid B_1 \leq b_1) \\
&\quad + \frac{1}{I-1}\frac{\hat{G}_1(b_1)}{\hat{g}_1(b_1)}\frac{\hat{m}_2^l(b_2 \mid b_1)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1)}\frac{\partial \hat{G}_2^l(b_2 \mid B_1 \leq b_1)}{\partial b_1}.
\end{aligned}$$

All the estimators that form $\hat{g}_{2|1}^l(b_2 \mid b_1)$ are known from 4.1-4.4 except $\frac{\partial}{\partial b_1}(\frac{\hat{m}_2^l(b_2|b_1)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1)})$. Since I know $\hat{m}_2^l(b_2 \mid b_1)$ and $\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1)$ from (26) and (27) I have the following.

$$\begin{aligned}
\frac{\partial}{\partial b_1}\left(\frac{\hat{m}_2^l(b_2 \mid b_1)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1)}\right) &= \frac{\sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \frac{\partial \lambda_\ell(b_1)}{\partial b_1}\frac{1}{h_2}K\left(\frac{b_2 - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1)\overline{K}_{2\ell}(b_2)} \\
& - \frac{\sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \lambda_\ell(b_1)\frac{1}{h_2}K\left(\frac{b_2 - B_{2\ell}^{max}}{h_2}\right)\sum_{\ell \in \mathcal{L}_I}\frac{\partial \lambda_\ell(b_1)}{\partial b_1}\overline{K}_{2\ell}(b_2)}{\left(\sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1)\overline{K}_{2\ell}(b_2)\right)^2},
\end{aligned}$$

where I know $\partial\lambda_\ell(b_1)/\partial b_1$ from 4.3. It implies that I have constructed $\hat{F}_1(v_1 \mid z, I)$, as all the estimators comprising $\hat{\xi}_1(b_1; z, I)$ are known and $\hat{G}_1(b_1 \mid z, I)$ is known from 4.2.

To construct the estimator of $F_{2|1}(\cdot \mid v_1, z, I)$, I use the following equality modified from (22) in 3.1.5.

$$\begin{aligned}\hat{\hat{F}}_{2|1}(\cdot \mid b_1, z, I) &\equiv \hat{\Pr}[V_2 \leq \cdot \mid B_1 = b_1, z, I] = \hat{\Pr}[V_2 \leq \cdot \mid \xi_1(B_1; z, I) = \xi_1(b_1; z, I), z, I] \\ &= \hat{\Pr}[V_2 \leq \cdot \mid V_1 = v_1, z, I] \equiv \hat{F}_{2|1}(\cdot \mid \underbrace{v_1}_{=\xi_1(b_1)}, z, I).\end{aligned}$$

It implies that $\hat{\hat{F}}_{2|1}(\cdot \mid b_1, z, I)$, which I know from 4.3, is equivalent to $\hat{F}_{2|1}(\cdot \mid v_1, z, I)$ given the increasing property of $\xi_1(\cdot; z, I)$. Since $\hat{\xi}_1(b_1; z, I)$ is a consistent estimator of $\xi_1(b_1; z, I)$, it follows that $\hat{\hat{F}}_{2|1}(\cdot \mid \hat{\xi}_1(b_1; z, I), z, I) = \hat{F}_{2|1}(\cdot \mid \hat{v}_1, z, I)$ is a consistent estimator of $\hat{F}_{2|1}(\cdot \mid v_1, z, I)$, which is also a consistent estimator of $F_{2|1}(\cdot \mid v_1, z, I)$. I can conclude that $\hat{\hat{F}}_{2|1}(\cdot \mid \hat{\xi}_1(b_1; z, I), z, I)$ is a consistent estimator of $F_{2|1}(\cdot \mid v_1, z, I)$.

To construct the estimator of $\delta(v_1, \cdot; z)$ I use $\hat{\hat{\delta}}(b_1, \cdot; z, I)$ from (34). I combine $\hat{\hat{\delta}}(b_1, \cdot; z, I)$ with the equivalence established in Appendix A.7,

$$\hat{\hat{\delta}}(b_1, \cdot; z, I) = \hat{\hat{\delta}}\left(s_1\left(v_1; z, I\right), \cdot; z, I\right) = \hat{\hat{\delta}}\left(\xi_1^{-1}\left(v_1; z, I\right), \cdot; z, I\right) = \hat{\delta}(\underbrace{v_1}_{=\xi_1(b_1)}, \cdot; z, I).$$

The second equality holds by the relationship $\xi_1(b_1; z, I) = s_1^{-1}(b_1; z, I) \Leftrightarrow \xi_1^{-1}(v_1; z, I) = s_1(v_1; z, I)$, as established in Theorem 3. Since $\hat{\xi}_1(b_1; z, I) = \hat{v}_1$ is a consistent estimator of $\xi_1(b_1; z, I) = v_1$, it follows that $\hat{\hat{\delta}}\left(\hat{\xi}_1\left(b_1; z, I\right), \cdot; z, I\right) = \hat{\delta}\left(\hat{v}_1, \cdot; z, I\right)$ is a consistent estimator of $\hat{\delta}\left(v_1, \cdot; z, I\right)$, which is also a consistent estimator of $\delta(v_1, \cdot; z, I)$. I can conclude that $\hat{\hat{\delta}}\left(\hat{\xi}_1\left(b_1; z, I\right), \cdot; z, I\right)$, which I have constructed in 4.4-4.5, is a consistent estimator of $\delta(v_1, \cdot; z, I)$. Using the idea from (35) and $\hat{\hat{\delta}}\left(\hat{\xi}_1\left(b_1; z, I\right), \cdot; z, I\right)$, I define a new estimator of $\delta(v_1, \cdot; z)$ as follows,

$$\hat{\delta}(v_1, \cdot; z) \equiv (\sum_{\tilde{I}=2}^{N} L_{\tilde{I}})^{-1} \sum_{I=2}^{N} L_I \hat{\hat{\delta}}\left(\hat{\xi}_1\left(b_1; z, I\right), \cdot; z, I\right).$$

The new estimator $\hat{\delta}(v_1, \cdot; z)$ is computed as a weighted average of $\hat{\hat{\delta}}\left(\hat{\xi}_1\left(b_1; z, I\right), \cdot; z, I\right)$. It implies that to construct $\hat{\delta}(v_1, \cdot; z)$, I need to have prior knowledge of both $\hat{\hat{\delta}}\left(b_1, \cdot; z, I\right)$ and $\hat{\xi}_1\left(b_1; z, I\right)$ for every $I \in \{2, \ldots, N\}$.

5 Monte Carlo Simulation

I evaluate the performance of our multi-step estimator by testing it on bid distributions that satisfy the assumptions in Theorem 3: they must be absolutely continuous, and [(2), (6), (10)] must be increasing and differentiable with respect to [b_2^w for any b_1, b_2^l for any b_1, b_1] — the following triplet satisfies the assumptions.

$$G^{w}_{2|1}(b_2 \mid b_1) = b_2^{2b_1}, \tag{36}$$

$$G^{l}_{2|1}(b_2 \mid b_1) = b_2^{b_1}, \tag{37}$$

$$G_1(b_1) = b_1, \tag{38}$$

where the supports are $b_1 \in [0,1] \equiv [\underline{b_1}, \overline{b_1}]$ and $b_2 \in [0,1] \equiv [\underline{b_2}, \overline{b_2}]$. Within the support $G^{w}_{2|1}(\cdot \mid b_1)$ first-order dominates $G^{l}_{2|1}(\cdot \mid b_1)$ for any given b_1, indicating that in the second auction the winner of the first auction likely bids higher than the loser of the first auction.

I assume a setting with three bidders($I = 3$) and no auction-specific covariates, Z. Given the triplet each bidder draws their first auction bid from $G_1(\cdot)$ and if a bidder i wins(resp., loses) he draws his second auction bid from $G^{w}_{2|1}(\cdot \mid b_{1i})$(resp., $G^{l}_{2|1}(\cdot \mid b_{1i})$). It generates a single auction pair, indexed by ℓ, which consists of $(B^{\max}_{1\ell}, W_{1\ell}, B^{\max}_{2\ell}, W_{2\ell}, \mathcal{I}_\ell = 3)$. Repeat the process 500 times resulting in a total of $L = 500$ auction pairs, i.e., $\ell \in \{1, \ldots, 500\}$. I observe that the winner of the first auction also wins the second auction in approximately $70\% - 75\%$ of the cases.

I define the sample as $\{(B^{\max}_{1\ell}, W_{1\ell}, B^{\max}_{2\ell}, W_{2\ell}, \mathcal{I}_\ell = 3) : \ell = 1, \ldots, 500\}$, and I create a total of 200 samples. Each sample yields a vector of estimates, $\left(\hat{G}^{w}_{2|1}(\cdot \mid \cdot), \ldots, \hat{\delta}(\cdot, \cdot)\right)$, as described in 4.1-4.5, so our 200 samples produce a total of 200 vectors of estimates. In its production, I chose the Gaussian function and Silverman's rule of thumb for kernel and bandwidths selection; to enhance computational speed in Python, I employ Numba and Multiprocessing.

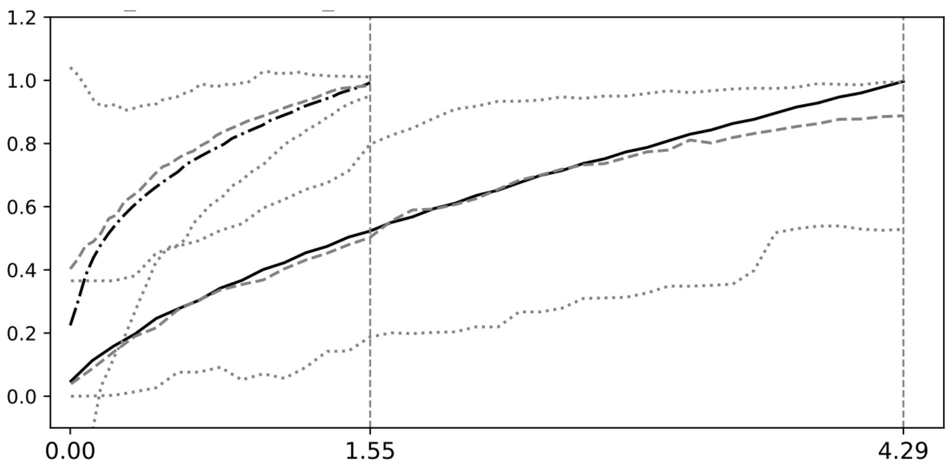

Figure 2: The solid line represents the true $\tilde{D}(d \mid 0.3)$ from (16), with d varying between $[\xi^{w}_2(0.3, \underline{b_2} = 0),\ \xi^{w}_2(0.3, \overline{b_2} = 1)] = [0,\ 4.29]$. The dash-dotted line represents the true $\tilde{F}_{2|1}(v_2 \mid 0.3)$ from (19), with v_2 varying between $[\xi^{l}_2(0.3, \underline{b_2} = 0),\ \xi^{l}_2(0.3, \overline{b_2} = 1)] = [0,\ 1.55]$. The dashed line and the dotted lines correspond to the (pointwise) 50% percentile and 80% confidence interval of 200 estimates, $\hat{\tilde{D}}(\cdot \mid 0.3)$ and $\hat{\tilde{F}}_{2|1}(\cdot \mid 0.3)$.

Set the first auction bid at $b_1 = 0.3$, and Figure 2 compares the estimates $\hat{\tilde{D}}(d \mid 0.3)$ and $\hat{\tilde{F}}_{2|1}(v_2 \mid 0.3)$, indicating that I am at stage 4.3 in 4.1-4.5. Both estimators are evaluated at 40 equally spaced points on $[\xi^{w}_2(0.3, \underline{b_2}),\ \xi^{w}_2(0.3, \overline{b_2})] = [0, 4.29]$ and $[\xi^{l}_2(0.3, \underline{b_2}),\ \xi^{l}_2(0.3, \overline{b_2})] = [0, 1.55]$. With a sample size of two hundred each grid point contains two hundred estimates, allowing us to construct pointwise 50% percentile(dashed) and 80% confidence interval(dotted). The dashed lines closely track the true lines(solid, dash-dotted) derived from (36)-(38).

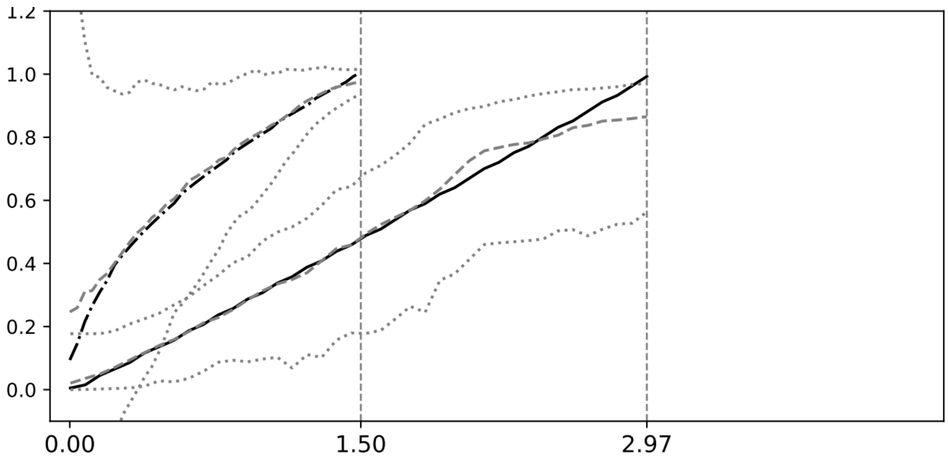

Figure 3: The solid line represents the true $\tilde{D}(d \mid 0.5)$ from (16), with d varying between $[\xi_2^w(0.5, \underline{b_2} = 0),\, \xi_2^w(0.5, \overline{b_2} = 1)] = [0,\, 2.97]$. The dash-dotted line represents the true $\tilde{F}_{2|1}(v_2 \mid 0.5)$ from (19), with v_2 varying between $[\xi_2^l(0.5, \underline{b_2} = 0),\, \xi_2^l(0.5, \overline{b_2} = 1)] = [0,\, 1.50]$. The dashed line and the dotted lines correspond to the (pointwise) 50% percentile and 80% confidence interval of 200 estimates, $\hat{\tilde{D}}(\cdot \mid 0.5)$ and $\hat{\tilde{F}}_{2|1}(\cdot \mid 0.5)$.

Figure 3 provides a comparison to Figure 2, with the only difference being an increase in the first auction bid from 0.3 to 0.5. The increase in b_1 leads to a reduction in the domains of v_2 and d. It implies that if the three auction bids were $[b_{1i} = 0.5, b_{1j} = 0.3, b_{1k} = 0.3]$ in a given ℓ-th auction pair, the maximum possible value of the second object for bidder i is 2.97 while for bidders $\{j, k\}$ it is 1.55. Figures 4 and 5 depict the estimated strategies for i and $\{j, k\}$.

It is unclear why a bidder who placed a higher bid in the first auction$(0.3 \to 0.5)$ perceives the second object v_2 as less valuable$(1.55 \to 1.50)$. I suspect that the phenomenon occurs because our model assumes that no bidders drop out within an auction pair, and the triplet (36)-(38) satisfy the assumption (Assumption 1); never dropping out is demonstrated as a first auction loser favoring the second object more$(1.50 \to 1.55)$ as their first auction bid decreases$(0.5 \to 0.3)$.

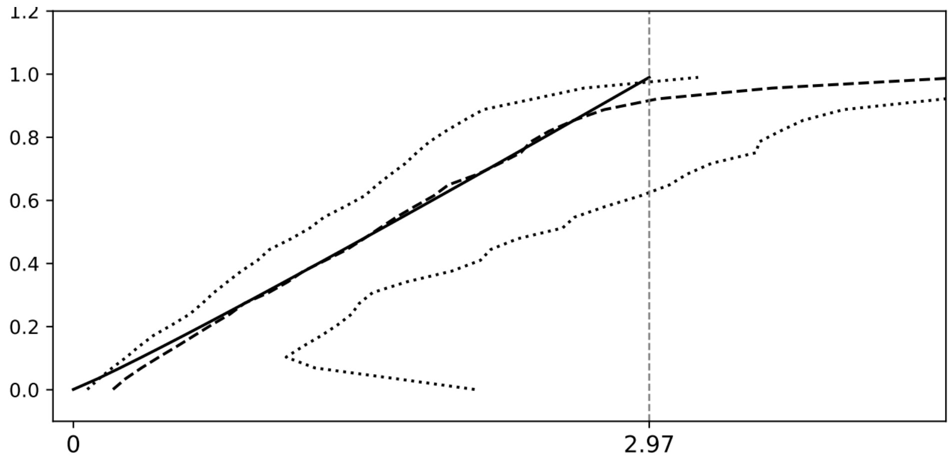

Figure 4: The X-axis represents $d \in [\xi_2^w(0.5, \underline{b_2} = 0), \xi_2^w(0.5, \overline{b_2} = 1)] = [0, 2.97]$, where $d \equiv \tilde{\delta}(0.5, V_2)$ defined in 3.1.3. The Y-axis represents $b_2^w \in [\underline{b_2} = 0, \overline{b_2} = 1]$. The plot illustrates the second auction equilibrium strategy for a first auction winner with an initial bid of 0.5. Let $v_1 = \xi_1^{-1}(0.5)$, then the solid line represents the true $s_2^w(v_1, V_2) \equiv s_2^w(v_1, \tilde{\delta}(0.5, V_2)) = (\xi_2^w)^{-1}(\tilde{\delta}(0.5, V_2); v_1)$ defined in Theorem 3. The dashed and the dotted lines correspond to the (pointwise) 50% percentile and 80% confidence interval of 200 estimates, $\hat{s}_2^w(v_1, V_2)$.

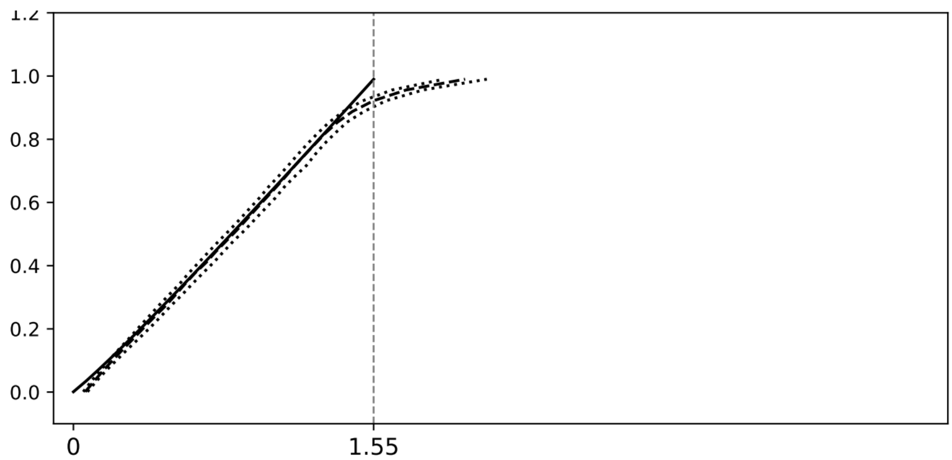

Figure 5: The X-axis and Y-axis represent $v_2 \in [\xi_2^l(0.3, \underline{b_2} = 0), \xi_2^l(0.3, \overline{b_2} = 1)] = [0, 1.55]$ and $b_2^l \in [\underline{b_2} = 0, \overline{b_2} = 1]$. The plot illustrates the second auction equilibrium strategy for a first auction loser with an initial bid of 0.3. Let $v_1 = \xi_1^{-1}(0.3)$, then the solid line represents the true $s_2^l(v_1, V_2) = (\xi_2^l)^{-1}(V_2; v_1)$ defined in Theorem 3. The dashed and the dotted lines correspond to the (pointwise) 50% percentile and 80% confidence interval of 200 estimates, $\hat{s}_2^l(v_1, V_2)$.

Figures 4 and 5 depict equilibrium strategies defined in Theorem 3, comparable to Figure 1 in Guerre et al. (2000). Theorem 3 establishes that the quasi-inverse bidding strategies for i and $\{j, k\}$ correspond to $\xi_2^w(0.5, b_2^w)$ from (2) and $\xi_2^l(0.3, b_2^l)$ from (6). Using the inverse bidding strategies the figures are generated by evaluating both $\hat{\xi}_2^w(0.5, b_2^w)$ and $\hat{\xi}_2^l(0.3, b_2^l)$ at 30 equally spaced points on the Y-axis, $[\underline{b_2}, \overline{b_2}] = [0, 1]$.

It is unclear why the confidence interval for $\hat{\xi}_2^w(0.5, b_2^w)$ in Figure 4 is larger than that for $\hat{\xi}_2^l(0.3, b_2^l)$ in Figure 5; I suspect that the phenomenon happens because of the triplet used in our simulation, (36)-(38). I observed that the winner of the first auction also wins the second auction in approximately 70% − 75% of the cases. Since I am in Case 1(3.1), it implies that the 70% − 75% and 25% − 30% of auction pairs correspond to $\{\ell \in \mathcal{L}_{I=3} : W_{1\ell} = W_{2\ell}\}$ and $\{\ell \in \mathcal{L}_{I=3} : W_{1\ell} \neq W_{2\ell}\}$, which together constitute the entire set $\{\ell \in \mathcal{L}_I\}$. Among the three sets the following equations show that $\hat{\xi}_2^w(0.5, b_2^w)$ never utilizes $\{\ell \in \mathcal{L}_{I=3} : W_{1\ell} = W_{2\ell}\}$, while $\hat{\xi}_2^l(0.3, b_2^l)$ incorporates information from all three sets.

$$\hat{\xi}_2^w(0.5, b_2^w) \equiv b_2^w + \frac{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2^w \mid 0.5)}{\hat{m}_2^l(b_2^w \mid 0.5)},$$
$$\hat{\xi}_2^l(0.3, b_2^l) \equiv b_2^l + \frac{\int_{0.3}^1 \hat{G}_2^l\left(b_2^l \mid B_1 \leq x\right)^{3-2} \hat{G}_{2|1}^w(b_2^l \mid x) d\hat{G}_1(x)^{3-1}}{\int_{0.3}^1 \hat{\Psi}(b_2^l; x)\hat{G}_2^l\left(b_2^l \mid B_1 \leq x\right)^{3-2} \hat{G}_{2|1}^w(b_2^l \mid x) d\hat{G}_1(x)^{3-1}},$$

where the equations come from (29) and (32), where I discarded the conditions $(Z = z, \mathcal{I} = I)$. I suspect that the difference in the amount of information utilized by the two estimators accounts for the disparity in confidence intervals.

Lastly, the estimate of a synergy function is depicted in Figure 6.

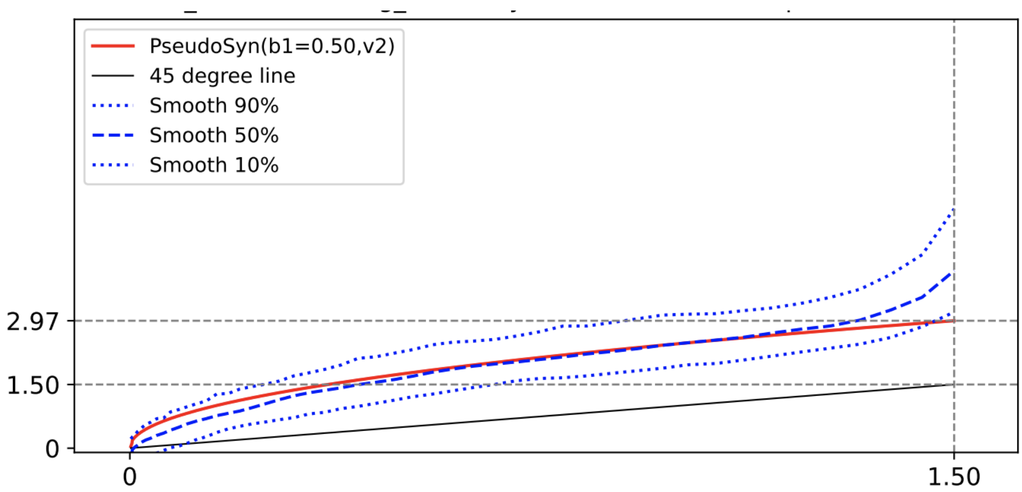

Figure 6: The X-axis and Y-axis represent $v_2 \in [\xi_2^l(0.5, \underline{b_2} = 0), \xi_2^l(0.5, \overline{b_2} = 1)] = [0, 1.50]$ and d varying between $[\xi_2^w(0.5, \underline{b_2} = 0), \xi_2^w(0.5, \overline{b_2} = 1)] = [0, 2.97]$. The plot illustrates a function $\tilde{\delta}(b_1 = 0.5, v_2)$.

I see that the setting of this Monte Carlo Simulation implies the case of positive synergy; moreover, since $\tilde{\delta}(b_1, v_2) = \delta(\xi_1(b_1), v_2)$ holds, the figure above shows the graph of a function δ where v_1 is fixed at $\xi_1(0.5)$.

6 Conclusion

I examined two-period first-price sealed-bid auction, where the auctioneer only discloses the winner's identity between the two auctions. Given the setting, I constructed the bidders' profit functions using the unobserved value as the dependent variable and the observed bids as explanatory variables. The approach, along with the distribution of V_2 being influenced by v_1, separates synergy and affiliation in our model. Based on the separation, I demonstrated that the analyst could identify synergy and affiliation separately even with limited observations, i.e., access to only maximum bids and winner's identities. Multi-step estimator followed the identification steps, enabling the analyst to estimate the degree of synergy and the level of affiliation between V_2 and V_1. I validated the performance of our estimator through Monte Carlo simulations, showing its reliability; our modeling approach allowed us to avoid computational burdens, leading us to test the estimator.

Throughout the paper I have assumed that the auctioneer only discloses the identity of the winner to the bidders after the first auction or at *step*(ii). He may provide additional information, such as the winning bid or all the bids, as discussed in Bergemann and Hörner (2018). The result of their paper is not applicable to our model for various reasons, such as the difference in equilibrium strategies. I plan to examine the case where the auctioneer discloses both the winning bid and the winner's identity after the first auction. Papers exist which examine the impact of more information disclosure on various aspects, such as pooling behavior(Bergemann and Hörner (2018)), allocative efficiency or the expected revenue(Dufwenberg and Gneezy (2002), Kannan (2012), Azacis (2020)). I plan to contribute to the topic in our future research.

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Appendices

A Proofs and Detail

A.1 Assumption 5

Detail: I relate *steps*(0)-(iii) to Assumption 5. Without loss of generality, let $I_\ell = 2$ where the set of bidders is $\{i, j\}$.

Step(i) implies that each i and j separately draws v_{1i} and v_{1j} from one of the model primitives, $F_1(\cdot)$. So, *step*(i) is related to V_{1i}, V_{1j} being independent and identically distributed from $F_1(\cdot)$ in Assumption 5.

Given that v_{1i} and v_{1j} are fixed at *step*(i), *step*(iii) implies that each i and j separately draws v_{2i} and v_{2j} from the model primitive, $F_{2|1}(\cdot|v_{1i})$ and $F_{2|1}(\cdot|v_{1j})$. This implies the following equality.

$$\Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i} | V_{1j} = v_{1j}, V_{2j} = v_{2j}] = \frac{\Pr[v_{1i}, v_{2i}, v_{2j} | v_{1j}]}{\Pr[v_{2j}|v_{1j}]} \\ = \frac{\Pr[v_{1i}, v_{2i}|v_{1j}]\Pr[v_{2j}|v_{1j}]}{\Pr[v_{2j}|v_{1j}]} \\ = \Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i}]. \tag{39}$$

The second equality holds by $(V_{1i}, V_{2i}) \perp V_{2j}|V_{1j}$; the conditional independence holds because the *steps*(i) and (iii) jointly imply $V_{2i} \perp V_{2j}|V_{1j}$ and $V_{1i} \perp V_{2j}|V_{1j}$. The last equality holds by $(V_{1i}, V_{2i}) \perp V_{1j}$. (39) is equivalent to the following equation.

$$f(v_{1i}, v_{2i}, v_{1j}, v_{2j}) = f(v_{1i}, v_{2i})f(v_{1j}, v_{2j}). \tag{40}$$

As a result, the pairs (V_{1i}, V_{2i}) and (V_{1j}, V_{2j}) are independent with the joint density shown in the left-hand side of (40).

A.2 Remark 1

Detail: Assumption 5 implies that $(s_1(V_{1j}), j = 1, \ldots, I_\ell)$ are independent and identically distributed from $\Pr[s_1(V_1) \leq \cdot]$. But, the second auction bids are not necessarily independent; without loss of generality, let $I_\ell = 2$ where the set of bidders is $\{i, j\}$. Then, in equilibrium, any second auction bid can be expressed as follows.

$$B_{2i} = s_2^w(V_{1i}, V_{2i})\mathbb{1}(W_1 = i) + s_2^l(V_{1i}, V_{2i})\mathbb{1}(W_1 \neq i) \\ = s_2^w(V_{1i}, V_{2i})\mathbb{1}(V_{1i} \geq V_{1j}) + s_2^l(V_{1i}, V_{2i})(1 - \mathbb{1}(V_{1i} \geq V_{1j})), \tag{41}$$

$$B_{2j} = s_2^w(V_{1j}, V_{2j})(1 - \mathbb{1}(V_{1i} \geq V_{1j})) + s_2^l(V_{1j}, V_{2j})\mathbb{1}(V_{1i} \geq V_{1j}). \tag{42}$$

W_1 records the index of the winner in the first auction. The last equality of (41) holds because Theorem 3 asserts that $s_1(\cdot)$ is an increasing strategy. Given (41) and (42), I want to show $\Pr[B_{2i} = b_{2i}|B_{2j} = b_{2j}] = \Pr[B_{2i} = b_{2i}]$, which is equivalent to proving the following.

$$\Pr\left(\text{Function of } (V_{1i}, V_{2i}, V_{1j}) \,\middle|\, \text{Function of } (V_{1j}, V_{2j}, V_{1i})\right) \\ = \Pr\left(\text{Function of } (V_{1i}, V_{2i}, V_{1j})\right). \tag{43}$$

Equality does not necessarily hold because Assumption 5 pertains to the independence of the pairs (V_{1i}, V_{2i}) and (V_{1j}, V_{2j}), rather than the pairs (V_{1i}, V_{2i}, V_{1j}) and (V_{1j}, V_{2j}, V_{1i}). Consequently, the second auction bids are not guaranteed to be independent; intuitively, the occurrence of the event $\mathbb{1}(W_1 = i)$ introduces correlation among bidders $\{i, j\}$.

A.3 Lemma 1

Proof of Lemma 1. Recall that a bidder i is the first auction winner without loss of generality. First, I show that $\{V_{2i}\}$ and $\{V_{1j}, V_{2j}\}, j \neq i$ are I_ℓ independent sets of random variables given

$\{V_{1,-i}^{\max} \leq V_{1i} = v_{1i}\}$, which is equivalent to proving the following equation.

$$
\begin{aligned}
& f\left(v_{11}, v_{21}, \ldots, v_{2i}, \ldots, v_{1I_\ell}, v_{2I_\ell} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}\right) \\
& = f(v_{2i} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}) \prod_{k \neq i} f(v_{1k}, v_{2k} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}).
\end{aligned} \tag{44}
$$

I will use the fact that $\{V_{1,-i}^{\max} \leq V_{1i} = v_{1i}\}$ and $\{V_{1i} = v_{1i}, V_{1k} \leq v_{1i}, k \neq i\}$ are equivalent. Then, the left-hand side of (44) is equivalent to the following equation.

$$
\frac{\Pr\left((V_{1i} = v_{1i}, V_{2i} = v_{2i}), (V_{1k} = v_{1k}, V_{2k} = v_{2k}, V_{1k} \leq v_{1i}, k \neq i)\right)}{\Pr[V_{1i} = v_{1i}, V_{1k} \leq v_{1i}, k \neq i]}. \tag{45}
$$

By Assumption 5, V_1 is independent across bidders, and a pair (V_1, V_2) is also independent across bidders. Also, v_{1i} is an arbitrarily chosen value. Thus, (45) equals the following.

$$
\begin{aligned}
& \frac{\Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i}] \prod_{k \neq i} \Pr[V_{1k} = v_{1k}, V_{2k} = v_{2k}, V_{1k} \leq v_{1i}]}{\Pr[V_{1i} = v_{1i}] \prod_{k \neq i} \Pr[V_{1k} \leq v_{1k}]} \\
& = f(v_{2i} | v_{1i}) \prod_{k \neq i} f(v_{1k}, v_{2k} | V_{1k} \leq v_{1i}) \\
& = f(v_{2i} | V_{1i} = v_{1i}, V_{1k} \leq v_{1i}, k \neq i) \prod_{k \neq i} f(v_{1k}, v_{2k} | V_{1i} = v_{1i}, V_{1j} \leq v_{1i}, j \neq i) \\
& = f(v_{2i} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}) \prod_{k \neq i} f(v_{1k}, v_{2k} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}).
\end{aligned} \tag{46}
$$

The second equality of (46) holds by the following two equations (47) and (48) — they use the independence property from Assumption 5. An arbitrary bidder k in (48) comes from $k \in \{1, \ldots, I_\ell\}/\{i\}$

$$
\begin{aligned}
& f(v_{2i} | V_{1i} = v_{1i}, V_{1k} \leq v_{1i}, k \neq i) \\
& = \frac{\Pr\left((V_{1i} = v_{1i}, V_{2i} = v_{2i}), V_{1k} \leq v_{1i}, k \neq i\right)}{\Pr[V_{1i} = v_{1i}, V_{1k} \leq v_{1i}, k \neq i]} = \frac{f(v_{1i}, v_{2i}) \prod_{k \neq i} F_1(v_{1i})}{f(v_{1i}) \prod_{k \neq i} F_1(v_{1i})} = f(v_{2i} | v_{1i}),
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
& f(v_{1k}, v_{2k} | V_{1i} = v_{1i}, V_{1j} \leq v_{1i}, j \neq i) \\
& = \frac{\Pr[(V_{1k} = v_{1k}, V_{2k} = v_{2k}, V_{1k} \leq v_{1i}), V_{1i} = v_{1i}, V_{1j} \leq v_{1i}, j \neq \{i, k\}]}{\Pr[V_{1i} = v_{1i}, V_{1j} \leq v_{1i}, j \neq i]} \\
& = \frac{\Pr[V_{1k} = v_{1k}, V_{2k} = v_{2k}, V_{1k} \leq v_{1i}] f(v_{1i}) \prod_{j \neq \{i,k\}} F_1(v_{1i})}{f(v_{1i}) \prod_{j \neq i} F_1(v_{1i})} = f(v_{1k}, v_{2k} | V_{1k} \leq v_{1i}).
\end{aligned} \tag{48}
$$

As (46) is the left-hand side of (44), I proved that the (44) is true.

Second, I transform (44) into the second auction bids. In equilibrium, the second auction bids for $j \in \{1, \ldots, I_\ell\}$ will satisfy the following.

$$
\begin{aligned}
B_{2j} & = s_2^w(V_{1j}, V_{2j}) \mathbb{1}(B_{1,-j}^{\max} \leq B_{1j}) + s_2^l(V_{1j}, V_{2j}) \mathbb{1}(B_{1,-j}^{\max} > B_{1j}) \\
& = s_2^w(V_{1j}, V_{2j}) \mathbb{1}(V_{1,-j}^{\max} \leq V_{1j}) + s_2^l(V_{1j}, V_{2j}) \mathbb{1}(V_{1,-j}^{\max} > V_{1j}),
\end{aligned} \tag{49}
$$

where the last equality holds because $s_1(\cdot)$ is an increasing function by Theorem 3. Then, the second auction equilibrium bid for each $j \in \{1, \ldots, I_\ell\}/\{i\}$ and i, given $\{V_{1,-i}^{\max} = V_{1i} \leq v_{1i}\}$, is as follows.

$$B_{2i} = s_2^w(v_{1i}, V_{2i}),$$
$$B_{2j} = s_2^l(V_{1j}, V_{2j}).$$

Now, note the three following facts where (a) by Theorem 3, $s_2^w(\cdot,\cdot), s_2^l(\cdot,\cdot)$ are measurable functions; (b) If $X, Y_1 \ldots, Y_n$ are mutually independent, then so are $g(X), h(Y_1), \ldots, h(Y_n)$ mutually independent, where $g(\cdot)$ and $h(\cdot)$ are measurable functions; and (c) v_{1i} is a fixed nonstochastic number. Given (a), (b), and (c), think of X and $Y_1, \ldots, Y_n$ in (b) as V_{2i} and $(V_{1j}, V_{2j}), j \neq i$, and also consider $g(\cdot)$ and $h(\cdot)$ as $s_2^w(v_{1i}, \cdot)$ and $s_2^l(\cdot,\cdot)$. Then, (44) equals the following.

$$\begin{aligned} &\Pr[B_{21} = b_{21}, \ldots, B_{2i} = b_{2i}, \ldots, B_{2I_\ell} = b_{2I_\ell} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}] \\ &= \Pr[B_{2i} = b_{2i} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}] \prod_{j \neq i} \Pr[B_{2j} = b_{2j} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}]. \end{aligned} \tag{50}$$

Note that the conditioning event $\{V_{1,-i}^{\max} \leq V_{1i} = v_{1i}\}$ in the left-hand side of the equation is the same as the event $\{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\}$ since $s_1(\cdot)$ is increasing. Also, the right-hand side of the equation is the same as follows.

$$\begin{aligned} &\Pr[B_{2i} = b_{2i} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}] \prod_{j \neq i} \Pr[B_{2j} = b_{2j} | V_{1,-i}^{\max} \leq V_{1i} = v_{1i}] \\ &= \Pr[B_{2i}^w = b_{2i} | V_{1i} = v_{1i}] \prod_{j \neq i} \Pr[B_{2j}^l = b_{2j} | V_{1j} \leq v_{1i}] \\ &= \Pr[B_{2i}^w = b_{2i} | B_{1i} = b_{1i}] \prod_{j \neq i} \Pr[B_{2j}^l = b_{2j} | B_{1j} \leq b_{1i}], \end{aligned}$$

where the first equality holds by (47) and (48) and the fact that i is the winner($B_{2i} = B_{2i}^w$) and $j \neq i$ are losers($B_{2j} = B_{2j}^l$). The second equality holds by the increasing $s_1(\cdot)$. Thus, (50) equals the following.

$$\begin{aligned} &\Pr[B_{21} = b_{21}, \ldots, B_{2i} = b_{2i}, \ldots, B_{2I_\ell} = b_{2I_\ell} | B_{1,-i}^{\max} \leq B_{1i} = b_{1i}] \\ &= \Pr[B_{2i}^w = b_{2i} | B_{1i} = b_{1i}] \prod_{j \neq i} \Pr[B_{2j}^l = b_{2j} | B_{1j} \leq b_{1i}]. \end{aligned} \tag{51}$$

As a result, I showed that $(B_{2i}, B_{2j}, j \neq i)$ are independent given $\{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\}$, and the distribution of B_{2i} given $\{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\}$ is $G_{2|1}^w(\cdot|b_{1i})$, whereas for $j \neq i$, the distribution of B_{2j} given $\{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\}$ is $G_2^l(\cdot|B_1 \leq b_{1i})$. ■

A.4 Comprehensive derivations of (4) and (8)

Since $\mathcal{V}^w(v_{1i}, b_{1i})$ is (3) I differentiate $\tilde{\pi}_2^w(v_{1i}, v_{2i}, b_{1i})$ with respect to b_{1i}.

$$\frac{\partial \tilde{\pi}_2^w(v_{1i}, v_{2i}, b_{1i})}{\partial b_{1i}} = \frac{\partial \pi_2^w\left(v_{1i}, v_{2i}, b_{1i}, \tilde{b}_{2i}^w\right)}{\partial b_{1i}} = \frac{H_2^w\left(\tilde{b}_{2i}^w; b_{1i}\right)}{h_2^w\left(\tilde{b}_{2i}^w; b_{1i}\right)} \frac{\partial H_2^w\left(\tilde{b}_{2i}^w; b_{1i}\right)}{\partial b_{1i}},$$

where the Envelope Theorem is used. From (1) I know the partial derivative of $H_2^w(\cdot; b_{1i})$ with respect to b_{1i},

$$\frac{\partial H_2^w(\cdot; b_{1i})}{\partial b_{1i}} = \frac{dG_1(b_{1i})^{I-1}/db_{1i}}{G_1(b_{1i})^{I-1}} \left[G_2^l(\cdot \mid B_1 \le b_{1i})^{I-2} G_{2|1}^l(\cdot \mid b_{1i}) - H_2^w(\cdot; b_{1i})\right].$$

Using $\partial \tilde{\pi}_2^w(v_{1i}, v_{2i}, b_{1i})/\partial b_{1i}$ the partial derivative of (3) with respect to b_{1i} yields (4).

$$\frac{\partial \mathcal{V}^w(v_{1i}, b_{1i})}{\partial b_{1i}} = \frac{dG_1(b_{1i})^{I-1}/db_{1i}}{G_1(b_{1i})^{I-1}} \times$$
$$\mathbb{E}_{V_2|V_1}\left[\frac{H_2^w(\tilde{B}_{2i}^w; b_{1i})}{h_2^w(\tilde{B}_{2i}^w; b_{1i})}\left[G_2^l(\tilde{B}_{2i}^w \mid B_1 \le b_{1i})^{I-2} G_{2|1}^l(\tilde{B}_{2i}^w \mid b_{1i}) - H_2^w(\tilde{B}_{2i}^w; b_{1i})\right] \mid v_{1i}\right].$$

Since $\mathcal{V}^l(v_{1i}, b_{1i})$ is (7) I differentiate $\tilde{\pi}_2^l(v_{2i}, b_{1i})$ with respect to b_{1i}.

$$\frac{\partial \tilde{\pi}_2^l(v_{2i}, b_{1i})}{\partial b_{1i}} = \frac{\partial \pi_2^l\left(v_{2i}, b_{1i}, \tilde{b}_{2i}^l\right)}{\partial b_{1i}} = \frac{H_2^l\left(\tilde{b}_{2i}^l; b_{1i}\right)}{h_2^l\left(\tilde{b}_{2i}^l; b_{1i}\right)} \frac{\partial H_2^l\left(\tilde{b}_{2i}^l; b_{1i}\right)}{\partial b_{1i}},$$

where the Envelope theorem is used. From (5) I know the partial derivative of $H_2^l(\cdot; b_{1i})$ with respect to b_{1i},

$$\frac{\partial H_2^l(\cdot; b_{1i})}{\partial b_{1i}} = \frac{dG_1(b_{1i})^{I-1}/db_{1i}}{1 - G_1(b_{1i})^{I-1}} \left[H_2^l(\cdot; b_{1i}) - G_2^l(\cdot \mid B_1 \le b_{1i})^{I-2} G_{2|1}^w(\cdot \mid b_{1i})\right].$$

Using $\partial \tilde{\pi}_2^l(v_{2i}, b_{1i})/\partial b_{1i}$ the partial derivative of (7) with respect to b_{1i} yields (8).

$$\frac{\partial \mathcal{V}^l(v_{1i}, b_{1i})}{\partial b_{1i}} = \frac{dG_1(b_{1i})^{I-1}/db_{1i}}{1 - G_1(b_{1i})^{I-1}} \times$$
$$\mathbb{E}_{V_2|V_1}\left[\frac{H_2^l(\tilde{B}_{2i}^l; b_{1i})}{h_2^l(\tilde{B}_{2i}^l; b_{1i})}\left[H_2^l(\tilde{B}_{2i}^l; b_{1i}) - G_2^l(\tilde{B}_{2i}^l \mid B_1 \le b_{1i})^{I-2} G_{2|1}^w(\tilde{B}_{2i}^l \mid b_{1i})\right] \mid v_{1i}\right].$$

A.5 Alternative derivation of $H_2^w(\cdot; b_{1i})$

An alternative derivation of (1) relies on noting that the distribution of $B_{2,-i}^{\max}$ in the second auction given $\left\{B_{1,-i}^{\max} = x, V_{1i} = v_{1i}, V_{2i} = v_{2i}\right\}$ for $\underline{b_1} \le x \le b_{1i}$ is $G_2^l(\cdot \mid B_1 \le x)^{I-2} G_{2|1}^l(\cdot \mid x)$ following Kong (2021). Using the fact, the distribution of $B_{2,-i}^{\max}$ given $\{B_{1,-i}^{\max} \le b_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}\}$ is

$$H_2^w(\cdot; b_{1i}) = \frac{1}{G_1(b_{1i})^{I-1}} \int_{\underline{b_1}}^{b_{1i}} G_2^l(\cdot \mid B_1 \le x)^{I-2} G_{2|1}^l(\cdot \mid x) dG_1(x)^{I-1}.$$

Hence, (1) is obtained by noting that the integrand is $\frac{d}{dx}\left[\int_{\underline{b_1}}^{x} G_{2|1}^l(\cdot \mid u) dG_1(u)\right]^{I-1}$.

A.6 Lemma 4

From Remarks 7.3.1 in Rao (1992), I know that the 'identified maximum' vector (Z, J), where $Z \equiv \max\{X_1, \ldots, X_k\}$ and $X_J = Z$, identifies the distributions $F_1(\cdot), \ldots, F_k(\cdot)$ of $X_1, \ldots, X_k$ when X_1 through X_k are mutually independent random variables with continuous distribution functions. Following the proof of Theorem 7.3.1 in Rao (1992), the next Lemma gives an explicit expression for $F_j(\cdot), j \in \{1, \ldots, k\}$.

Lemma 4 *Let $X_1, \ldots, X_k$ be mutually independent random variables with continuous distribution functions $F_1(\cdot), \ldots, F_k(\ldots)$. Define $Z \equiv \max\{X_1, \ldots, X_k\}$ and let J be the index such that $X_J = Z$, representing the random variable that achieves the maximum value. Given the vector (Z, J) the distributions $F_1(\cdot), \ldots, F_k(\cdot)$ are identified, where I have*

$$\begin{aligned} F_j(x) &= \exp\left\{ -\int_x^{+\infty} \left[\sum_{i=1}^k H_i(t) \right]^{-1} dH_j(t) \right\} \\ &= \exp\left\{ -\int_x^{+\infty} (\Pr[Z \le t])^{-1} dH_j(t) \right\}, \end{aligned} \tag{52}$$

where $H_j(x) \equiv \Pr[Z \le x, J = j]$ for $j = 1, \ldots, k$.

Proof of Lemma 4. Since $H_j(x) = \Pr[X_j$ is the maximum among $X_1, \ldots, X_k$, and $X_j \le x]$, I have

$$H_j(x) = \int_{-\infty}^x \prod_{i \ne j} F_i(t) dF_j(t) = \int_{-\infty}^x \frac{\prod_{i=1}^k F_i(t)}{F_j(t)} dF_j(t) = \int_{-\infty}^x \prod_{i=1}^k F_i(t) d\log F_j(t).$$

But, $\sum_{i=1}^k H_i(t) = \sum_{i=1}^k \Pr[Z \le t, J = i] = \Pr[Z \le t] = \prod_{i=1}^k F_i(t)$. Thus,

$$H_j(x) = \int_{-\infty}^x \sum_{i=1}^k H_i(t) d\log F_j(t).$$

Differentiating with respect to x gives

$$d\log F_j(x) = \left[\sum_{i=1}^k H_i(x) \right]^{-1} dH_j(x).$$

Integrating from x to $+\infty$ and noting that $\log F_j(+\infty) = 1$ gives

$$-\log F_j(x) = \int_x^{+\infty} \left[\sum_{i=1}^k H_i(t) \right]^{-1} dH_j(t),$$

which gives (52) since $\sum_{i=1}^k H_i(t) = \Pr[Z \le t]$. ■

A.7 Equivalence of $\tilde{\delta}(B_1, V_2)$ and $\delta(V_1, V_2)$

Choose an arbitrary value v_1 from the interval $[\underline{V_1}, \overline{V_1}]$, which fixes the domain of V_2 to $[\underline{V_2}, \overline{V_2}]$. The specific range $[\underline{V_2}, \overline{V_2}]$ may vary depending on the chosen v_1. Based on the definition of a function from Epp (2010), $\delta(V_1 = v_1, \cdot)$ implies two properties: (a) every element in $[\underline{V_2}, \overline{V_2}]$ is associated with an element in $\mathbb{R}_+$, and (b) no element in $[\underline{V_2}, \overline{V_2}]$ is associated with more than one element in $\mathbb{R}_+$. By Theorem 3, the function $s_1(\cdot)$ is increasing, guaranteeing the existence of a unique b_1 such that $s_1(v_1) = b_1$. Given b_1, the relation between $[\underline{V_2}, \overline{V_2}]$ and $\mathbb{R}_+$ remains unchanged. Therefore, I can define a new function $\tilde{\delta}(s_1(V_1) = b_1, \cdot) : [\underline{V_2}, \overline{V_2}] \to \mathbb{R}_+$, which is equivalent to $\delta(V_1 = v_1, \cdot) : [\underline{V_2}, \overline{V_2}] \to \mathbb{R}_+$. Since v_1 was arbitrarily chosen, I can vary v_1. Moreover, because of the increasing nature of $s_1(\cdot)$, distinct values of v_1 yield different b_1. As a result, I establish the equivalence $\delta(V_1, V_2) = \tilde{\delta}(s_1(V_1), V_2) \equiv \tilde{\delta}(B_1, V_2)$.